Some Facts about Idempotent Matrices

1. A square matrix $\mathbf{A}$ is *idempotent* if and only if $\mathbf{A}^2 = \mathbf{A}$.

2. If $\mathbf{A}$ is idempotent then $\text{trace}(\mathbf{A}) = \text{rank}(\mathbf{A})$.

3. $\mathbf{A}$ is idempotent if and only if $\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A}) = n$ where the dimensions of $\mathbf{A}$ are $n \times n$ and $\mathbf{I}$ is the $n \times n$ identity matrix.

4. The matrices $\mathbf{H}$, $\mathbf{I} - \mathbf{H}$, $\frac{1}{n}\mathbf{J}$, and $\mathbf{H} - \frac{1}{n}\mathbf{J}$ (as defined in lecture) are idempotent. (For the last of these, it is first necessary to show that $\mathbf{HJ} = \mathbf{J}$.)

5. If $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$ where $\mathbf{A}$, $\mathbf{A}_1$, $\mathbf{A}_2$ are all idempotent then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}_1) + \text{rank}(\mathbf{A}_2)$.