STA 302 / 1001

Note Title 11/1/2011

<u>Reminder</u> Andriy in RW 107 & 109 to help with Assignment 2

Today 11:00 – 1:00

Tues. 10 – 12

Wed Nov 9 10 – 12

Simple Linear in Matrix Form

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, \quad X = \begin{pmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \quad e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

Model: $Y = X\beta + e$

Least squares estimators: $\hat{\beta} = (X'X)^{-1}X'Y$

Your homework: Calculate $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{1}{nS_{xx}} \begin{bmatrix} \sum X_i^2 & -\sum X_i \\ -\sum X_i & n \end{bmatrix}$$

$$(X'X)^{-1}X'Y = \begin{bmatrix} \sum X_i^2 / nS_{xx} & -\bar{X}/S_{xx} \\ -\bar{X}/S_{xx} & 1/S_{xx} \end{bmatrix} \begin{bmatrix} \sum Y_i \\ \sum X_i Y_i \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \dfrac{(\sum X_i^2)\bar{Y}}{S_{xx}} - \dfrac{\bar{X}\sum X_i Y_i}{S_{xx}} \\ -\dfrac{n\bar{X}\bar{Y}}{S_{xx}} + \dfrac{\sum X_i Y_i}{S_{xx}} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$$

<u>Recall</u>: $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

<u>Exercise</u> show this = formula above

<u>Properties of Least Squares Estimators</u>

$$\hat{\beta} = (X'X)^{-1} X' Y$$

$$E(\hat{\beta} \mid X) = E\left((X'X)^{-1} X' Y \mid X \right)$$

$$= (X'X)^{-1} X' E(Y \mid X)$$

Model

$$\boxed{Y = X\beta + e}$$

$$= (X'X)^{-1} X' X \beta$$

$$= \beta \quad , \text{ unbiased.}$$

$$\mathrm{Var}(\hat{\beta} \mid X) = \mathrm{Var}\left((X'X)^{-1} X' Y \mid X\right)$$

$\mathrm{Var}(AX) = A\,\mathrm{Var}(X)\,A'$

$$= (X'X)^{-1} X' \,\mathrm{Var}(Y \mid X) \left[(X'X)^{-1} X' \right]'$$

$$= (X'X)^{-1} X' \sigma^2 I \, X (X'X)^{-1}$$

$\left[(X'X)^{-1}\right]' = (X'X)^{-1}$ because $(X'X)^{-1}$ is symmetric

$$= \sigma^2 (X'X)^{-1} \underbrace{X'X}\,\underbrace{(X'X)^{-1}}$$

$$\cancel{(X'X)^{-1} = X^{-1}(X')^{-1}}$$

$$= \sigma^2 (X'X)^{-1}$$

<u>Gives</u> $\text{Var}(\hat{\beta} \mid X) = \sigma^2 \begin{pmatrix} \dfrac{\sum x_i^2}{n S_{xx}} & -\dfrac{\bar{x}}{S_{xx}} \\ -\bar{x}/S_{xx} & \dfrac{1}{S_{xx}} \end{pmatrix}$.

<u>Exercise</u> show $\sigma^2 \dfrac{\sum x_i^2}{n S_{xx}}$ is equivalent to formula

for $\mathrm{Var}(\hat{\beta}_0 | X)$ that we had before.