# Supplement for "Nonparametric Covariate Adjustment for Receiver Operating Characteristic Curves"

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LEMMA 1 *If the assumptions (A1)–(A3) hold, and $m \to \infty$, for a given $z$,*

$$\sqrt{mh_1}\{\hat{\mu}_1(z) - \mu_1(z), \hat{v}_1(z) - v_1(z)\}^T \xrightarrow{D} N\{\boldsymbol{b}_1(z), \Sigma_1(z)\}, \tag{1}$$

*where $\boldsymbol{b}_1(z) = \{b_{11}(z), b_{12}(z)\}^T$ and $\Sigma_1(z) = \{\sigma_{x,ij}(z)\}_{1\le i,j\le 2}$ with*

$$b_{11}(z) = \frac{m_{p+1}(K^*)}{(p+1)!} d_1 \mu_1^{(p+1)}(z), \qquad b_{12}(z) = \frac{m_{p+1}(K^*)}{(p+1)!} d_1 \rho_1^{p+1} v_1^{(p+1)}(z),$$

$$\sigma_{x,11}(z) = \frac{R(K^*) v_1(z)}{\theta(z)}, \quad \sigma_{x,22}(z) = \frac{R(K^*)\kappa_1(z)}{\theta(z)\rho_1}, \quad \sigma_{x,12}(z) = \frac{R(K^*, \rho_1)\eta_1(z)}{\theta(z)\rho_1}.$$

*Analogously, if the assumptions (A1), (A4)–(A5) hold, and $n \to \infty$, for a given $z$,*

$$\sqrt{nh_2}\{\hat{\mu}_2(z) - \mu_2(z), \hat{v}_2(z) - v_2(z)\}^T \xrightarrow{D} N\{\boldsymbol{b}_2(z), \Sigma_2(z)\}, \tag{2}$$

*where $\boldsymbol{b}_2(z) = \{b_{21}(z), b_{22}(z)\}^T$ and $\Sigma_2(z) = \{\sigma_{y,ij}(z)\}_{1\le i,j\le 2}$ with*

$$b_{21}(z) = \frac{m_{p+1}(K^*)}{(p+1)!} d_2 \mu_2^{(p+1)}(z), \qquad b_{22}(z) = \frac{m_{p+1}(K^*)}{(p+1)!} d_2 \rho_2^{p+1} v_2^{(p+1)}(z),$$

$$\sigma_{y,11}(z) = \frac{R(K^*) v_2(z)}{\theta(z)}, \quad \sigma_{y,22}(z) = \frac{R(K^*)\kappa_2(z)}{\theta(z)\rho_2}, \quad \sigma_{y,12}(z) = \frac{R(K^*, \rho_2)\eta_2(z)}{\theta(z)\rho_2}.$$

*Proof of Lemma 1.* The asymptotic normality of $\hat{\mu}_1(z)$ with the bias $b_{11}$ and the variance $\sigma_{x,11}$ is standard in local polynomial regression. Let $v_{i,x}^* = \{x_i - \mu_1(z_{i,x})\}^2$, note that the input data $v_{i,x} = \{x_i - \hat{\mu}_1(z_{i,x})\}^2 = v_{i,x}^* + 2\{x_i - \mu_1(z_{i,x}\}\{\hat{\mu}_1\{z_{i,x} - \mu_1(z_{i,x})\} + \{\hat{\mu}_1(z_{i,x}) - \mu_1(z_{i,x})\}^2$. Applying a local polynomial fit to $(z_{i,x}, v_{i,x}), i = 1, \ldots, m$, one can see that the second term will result in a quantity of the order $o_p(b_1^{p+1} + 1/\sqrt{mb_1})$ and the third term will yield $O_p\{h_1^{2(p+1)} + 1/(mh_1)\}$. It is obvious that both quantities are ignorable, compared to the local polynomial estimator $v_1^*(z)$ obtained by fitting $(z_{i,x}, v_{i,x}^*)$. Therefore the estimators $\hat{v}_1(z)$ and $v_1^*(z)$ are asymptotically equivalent with the same limit distribution. Again we apply the standard argument of local polynomial regression to obtain the asymptotic normality of $\hat{v}_1(z)$ with the bias $b_{12}$ and variance $\sigma_{x,22}$. To derive the covariance of the limit distribution between $\hat{\mu}_1(z)$ and $\hat{v}_1(z)$, one can equivalently work with $\hat{\mu}_1(z)$ and $v^*(z)$. Using the equivalent kernel notation $K^*$, the limiting covariance is identical to the following, obtained by employing a Taylor expansion,

$$\text{cov}\{\tilde{\mu}_1(z) - \mu_1(z), \tilde{v}_1(z)\} = \frac{1}{mh_1\rho_1\theta(z)}\{\int K^*(u)K^*(u/\rho_1)du\eta_1(z) + O(h)\}.$$

where

$$\tilde{\mu}_1(z) = \frac{1}{mh_1\theta(z)} \sum_{i=1}^m K^*(\frac{z_{i,x} - z}{h_1}) x_i, \qquad \tilde{v}_1(z) = \frac{1}{mb_1\theta(z)} \sum_{i=1}^m K^*(\frac{z_{i,x} - z}{b_1}) v_i^*.$$

The same arguments can be applied to obtain the joint asymptotic distribution in Lemma 2.

LEMMA 2 *If the assumptions (A1$^\dagger$)-(A3$^\dagger$), (A6)–(A7) hold, and $m \to \infty$,*

$$\sup_{z \in \mathcal{Z}} |\hat{\mu}_1(z) - \mu_1(z)| = O(\tau_m), \qquad \sup_{z \in \mathcal{Z}} |\hat{v}_1(z) - v_1(z)| = O(\tau_m), \qquad w.p.1., \tag{3}$$

*and If the assumptions (A1$^\dagger$), (A4$^\dagger$)-(A5$^\dagger$), (A6) and (A8) hold, and $n \to \infty$,*

$$\sup_{z \in \mathcal{Z}} |\hat{\mu}_2(z) - \mu_2(z)| = O(\omega_n), \qquad \sup_{z \in \mathcal{Z}} |\hat{v}_2(z) - v_2(z)| = O(\omega_n), \qquad w.p.1, \tag{4}$$

*where $\tau_m = h_1^{p+1} + \sqrt{\log(1/h_1)/(mh_1)}$ and $\omega_n = h_2^{p+1} + \sqrt{\log(1/h_2)/(nh_2)}$ as defined in Theorem 2.*

*Proof of Lemma 2.* It is sufficient to show (3). The strong uniform convergence rate $\tau_m$ for $\hat{\mu}_1$ was obtained by Horng (2006), which is based on the arguments in Silverman (1978) and Mack and Silverman (1982) and the equivalent kernel representation, we follow the similar argument used in the proof of Lemma 1. Recall that $v_{i,x}^* = \{x_i - \mu_1(z_{i,x})\}^2$, and $v_{i,x} = \{x_i - \hat{\mu}_1(z_{i,x})\}^2 = v_{i,x}^* + 2\{x_i - \mu_1(z_{i,x}\}\{\hat{\mu}_1\{z_{i,x} - \mu_1(z_{i,x})\} + \{\hat{\mu}_1(z_{i,x}) - \mu_1(z_{i,x})\}^2$. Applying a local polynomial fit to $(z_{i,x}, v_{i,x})$, $i = 1, \ldots, m$, the second and third terms of the resulting estimator tend to 0 with probability 1, and the leading term has the strong uniform convergence rate $\tau_m$ by using the same argument for $\hat{\mu}_1$.