On the distribution of the sum of 12 independent uniform random variables

Introduction

In class, we noted that we could generate $\mathcal{N}(0,1)$ random variates using a sum of 12 independent $\text{Unif}(0,1)$ random variates $U_1, \cdots, U_{12}$:

$$\begin{aligned} S &= U_1 + \cdots + U_{12} - 6 \\ &= (U_1 - 1/2) + \cdots + (U_{12} - 1/2) \end{aligned}$$

In claiming that the distribution of S is approximately $\mathcal{N}(0,1)$, we are relying on the Central Limit Theorem as well as the fact that $E(S) = 0$ and $\text{Var}(S) = 1$ since $E(U_i) = 1/2$ and $\text{Var}(U_i) = 1/12$.

How close is the distribution of S to a $\mathcal{N}(0,1)$ distribution? It turns out that we can evaluate the exact distribution of S quite accurately using Fourier transforms and their inverses.

Fourier analysis

If $f(x)$ is a density function on the real-line, we can define its Fourier transform (also known as its characteristic function in probability theory) by

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} \exp(itx) f(x)\, dx \\ &= \int_{-\infty}^{\infty} \cos(tx) f(x)\, dx + i \int_{-\infty}^{\infty} \sin(tx) f(x)\, dx \end{aligned}$$

where $i = \sqrt{-1}$. Note that if $f(x)$ is symmetric around 0 then the imaginary component of $g(t)$ is 0 and so $g(t)$ is real-valued.

Given the Fourier transform $g(t)$, we can obtain the density function via the Fourier inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) g(t)\, dt.$$

In the case where $g(t)$ is real-valued (for all t), we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(tx) g(t)\, dt.$$

If $U_i \sim \text{Unif}(0,1)$ then $U_i - 1/2 \sim \text{Unif}(-1/2, 1/2)$ and so $U_i - 1/2$ has a symmetric density around 0. Its Fourier transform is given by

$$g_u(t) = \int_{-1/2}^{1/2} \exp(itx)\, dx$$

$$
\begin{aligned}
&= \int_{-1/2}^{1/2} \cos(tx)\,dx \\
&= \frac{2\sin(t/2)}{t}
\end{aligned}
$$

and the Fourier transform of $S = (U_1 - 1/2) + \cdots + (U_{12} - 1/2)$ is

$$g_s(t) = \left\{\frac{2\sin(t/2)}{t}\right\}^{12}$$

(using the independence of $U_1, \cdots, U_{12}$) and so the density of S is

$$
\begin{aligned}
f_s(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itx) g_s(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(tx) g_s(t)\,dt \\
&= \frac{1}{\pi}\int_{0}^{\infty} \cos(tx)\left\{\frac{2\sin(t/2)}{t}\right\}^{12} dt
\end{aligned}
$$

since $g_s(t)$ is real-valued and symmetric around 0.

Unfortunately, no closed-form expression for $f_s(x)$ exists and so we must compute $f_s(x)$ using numerical integration. In the next section, we will show how to do this using the numerical and symbolic software Maple. (We could also do the numerical integration in Matlab or R.)

Evaluation of $f_s(x)$ using Maple and R

We will evaluate $f_s(x)$ for $x = 0, 0.05, 0.10, \cdots, 6.00$ using the following Maple code:

```
> kernelopts(printbytes=false);
> f := t -> 2*sin(t/2)/t;
> g := y -> int(cos(y*t)*f(t)^12,t=0..infinity)/Pi;
> for y from 0 by 0.05 to 6
> do
> lprint(y,evalf(g(y)))
> od;
> quit;
```

(Note that $f_s(-x) = f_s(x)$.)

Putting the output from Maple into a file `sumofunifs.txt`, we can then use R to plot $f_s(x)$ and compare it to a $\mathcal{N}(0,1)$ density. (We could also do this in Maple itself.) The R code is as follows:

```
> r <- scan("sumofunifs.txt",what=list(0,0),sep=",")
> x <- r[[1]]
```

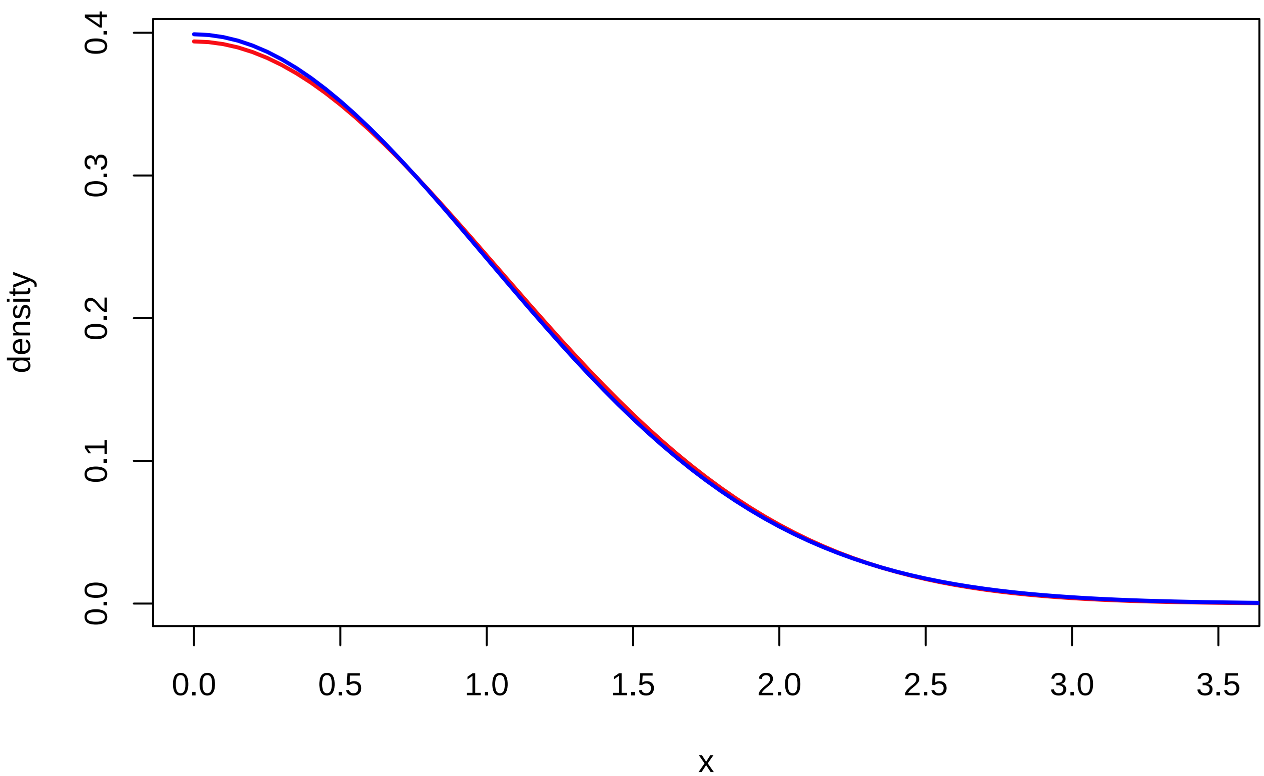

Figure 1: Density of S (red) and $\mathcal{N}(0, 1)$ density (blue).

```
> fs <- r[[2]]
> plot(x,fs,ylab="density",xlim=c(0,3.5),type="l",col="red",lwd=2)
> lines(x,dnorm(x),col="blue",lwd=2)
> plot(x,abs(fs-dnorm(x)),ylab="abs. diff.",xlim=c(0,3.5),
+ type="l",col="red",lwd=2)
```

Figure 1 compares the density $f_s(x)$ to the $\mathcal{N}(0, 1)$ density for $x \geq 0$; they are almost indistinguishable. Figure 2 gives a plot of $|f_s(x) - \phi(x)|$ versus x where $\phi(x)$ is the $\mathcal{N}(0, 1)$ density. (Note that the comparisons for $x \geq 0$ suffice since both $f_s(x)$ and $\phi(x)$ are symmetric around 0.)

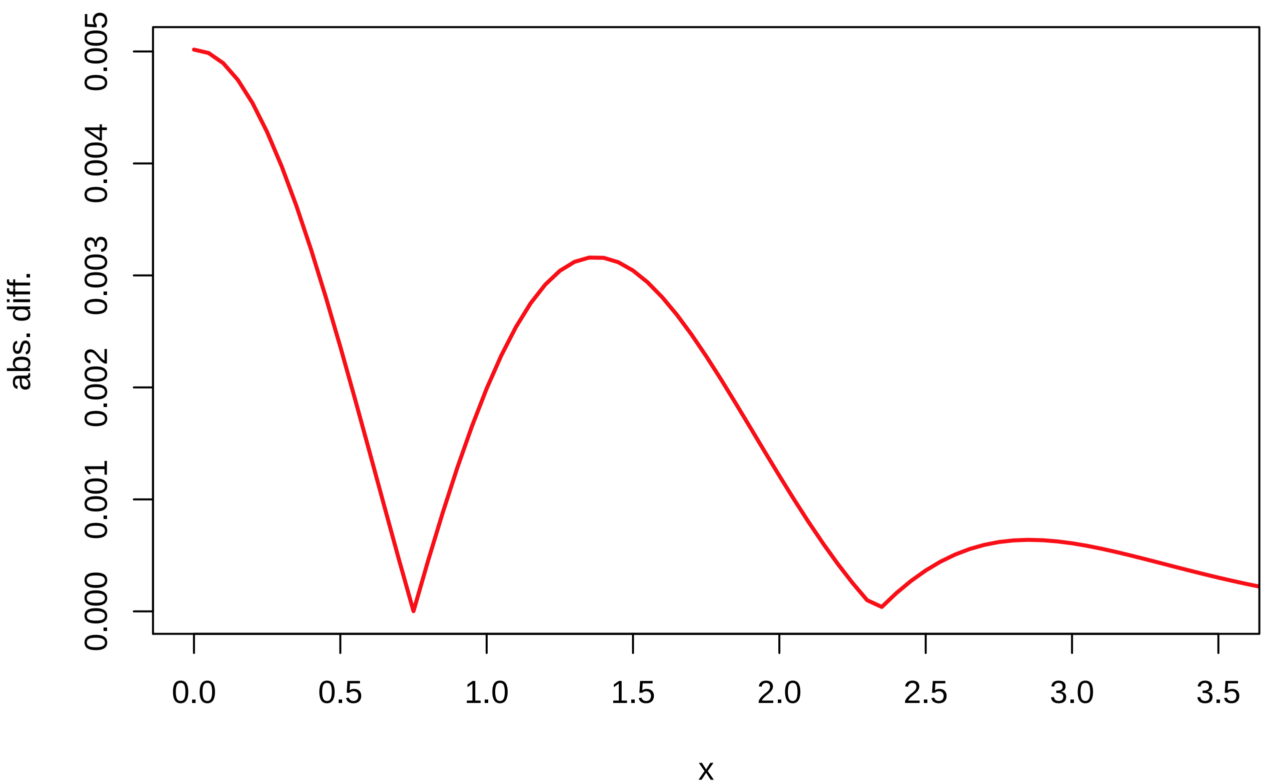

Figure 2: $|f_s(x) - \phi(x)|$ versus x where $\phi(x)$ is the $\mathcal{N}(0, 1)$ density.