Exercises 2 STA 3000, 2020

1. Suppose that $x = (x_1, \ldots, x_n) \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$ where $(\mu, \sigma^2) \in R^1 \times (0, \infty)$ is unknown.
(i) Show that $(\bar{x}, \sum_{i=1}^n (x_i - \bar{x})^2)$ is a mss and state its distribution.
(ii) Suppose the prior is given by

$$\begin{aligned} \mu \,|\, \sigma^2 &\sim N(\mu_0, \tau_0^2 \sigma^2), \\ \frac{1}{\sigma^2} &\sim \text{gamma}_{rate}(\alpha_0, \beta_0), \end{aligned}$$

where $\mu_0, \tau_0^2, \alpha_0, \beta_0$ are specified hyperparameters. Determine the posterior distribution of (μ, σ^2).
(iii) If $\Psi(\mu, \sigma^2) = \mu$ and $L((\mu, \sigma^2), \psi) = (\mu - \psi)^2$, determine the Bayes rule and the Bayes risk.
(iv) If $\Psi(\mu, \sigma^2) = \mu^3$ and $L((\mu, \sigma^2), \psi) = (\mu^3 - \psi)^2$, determine the Bayes rule. Are Bayes rules invariant under 1-1, smooth reparameterizations?
(v) If $\Psi(\mu, \sigma^2) = \sigma^2$ and $L((\mu, \sigma^2), \psi) = (\sigma^2 - \psi)^2 / \sigma^2$, determine the Bayes rule and the Bayes risk.

2. Suppose that $x = (x_1, \ldots, x_n) \overset{i.i.d.}{\sim} \text{Bernoulli}(\theta)$ where $\theta \in (0, 1)$ is unknown. Suppose $\Psi(\theta) = \theta$ and $L(\theta, \psi) = (\theta - \psi)^2$.
(i) Show that $\sum_{i=1}^n x_i$ is a mss and state its distribution.
(ii) Determine the supremum risk of $\bar{x}$.
(iii) Determine the risk function of the randomized estimator given by

$$\delta(\bar{x}, \{a\}) = \begin{cases} n/(n+1) & a = \bar{x} \\ 1/(n+1) & a = 1/2. \end{cases}$$

and use this to show $\bar{x}$ is not minimax.
(iv) With prior $\theta \sim \text{beta}(a, b)$ determine the posterior distribution of θ.
(v) Determine the Bayes rule implied by the prior in (iv).
(vi) For the Bayes rule in (v) determine its risk function and determine (a, b) so that the Bayes rule has constant risk.
(vii) Determine a minimax estimator. Is this estimator admissible? Is the associated prior least favourable?
(viii) Is the solution determined in (vii) unique?