Exercises 3 STA 3000, 2020

1. Suppose that $x = (x_1, \ldots, x_n) \overset{i.i.d.}{\sim} N(\mu, 1)$ where $\mu \in \{\mu_0, \mu_1\}$ where $\mu_0 < \mu_1$ and consider various approaches to assessing the hypotheis $H_0 = \mu_0$.
(a) Determine the likelihood ratio $\text{LR}(\bar{x}) = f_{\mu_1}(\bar{x})/f_{\mu_0}(\bar{x})$ and the probabilities $P_{\mu_i}(\text{LR}(\bar{x}) > 1)$ for $i = 0, 1$. Determine the limiting values as $n \to \infty$ of $\text{LR}(\bar{x})$ when H_0 is true and when it is false and the limiting values of $P_{\mu_i}(\text{LR}(\bar{x}) > 1)$ for $i = 0, 1$.
(b) Suppose the prior $\Pi(\{\mu_0\}) = p$ is used where $0 < p < 1$. Show that the Bayes factor $BF(\{\mu_1\} \,|\, \bar{x})$ in favor of μ_1 is given by $\text{LR}(\bar{x})$. Determine the limiting values of $\Pi(\{\mu_0\} \,|\, \bar{x})$ as $n \to \infty$ when H_0 is true and when it is false. Determine the limiting value of $RB(\{\mu_0\} \,|\, \bar{x}) = \Pi(\{\mu_0\} \,|\, \bar{x})/\Pi(\{\mu_0\})$ as $n \to \infty$ when H_0 is true and when it is false. Show that $BF(\{\mu_0\} \,|\, \bar{x}) > 1$ iff $RB(\{\mu_0\} \,|\, \bar{x}) > 1$.
(c) As shown in class the MP size α test is of the form reject H_0 when $\text{LR}(\bar{x}) > k_0(n, \mu_0, \mu_1, \alpha)$. Determine the limiting probabilities

$$\lim_{n \to \infty} P_{\mu_i}(k_0(n, \mu_0, \mu_1, \alpha) < LR(\bar{x}) \leq 1)$$

for $i = 0, 1$.
(d) The p-value approach is to consider a future independent sample with MSS $\bar{X}$, compute the tail probability

$$P_{\mu_0}(\sqrt{n}(\bar{X} - \mu_0) > \sqrt{n}(\bar{x} - \mu_0) \,|\, \bar{x})$$

and assert that evidence against H_0 has been obtained when this is small (typically less than some predetermined α like $\alpha = 0.05$). Determine the limiting values of this p-value as $n \to \infty$ when H_0 is true and when it is false.
(e) Comment on the degree to which these various approaches can give contradictory conclusions such as when the methods of parts (a) and (b) can indicate evidence in favor of H_0 while those of parts (c) and (d) suggest H_0 is false.

2. Suppose that $x = (x_1, \ldots, x_n) \overset{i.i.d.}{\sim}$ Bernoulli(p) where $p \in [0, 1]$ is unknown.
(a) If F denotes the cdf of the binomial(n, p) distribution, then prove (use integration by parts)

$$F(z) = \frac{\Gamma(n+1)}{\Gamma(z+1)\Gamma(n-z)} \int_\theta^1 y^z (1-y)^{n-z-1} \, dy.$$

(b) Determine the UMP size α test of $H_0 : p \leq p_0$ versus $H_a : p > p_0$.

3. Suppose that $x = (x_1, \ldots, x_n) \overset{i.i.d.}{\sim} N(\mu_0, \sigma^2)$ where $\sigma^2 > 0$ is unknown and μ_0 is known. Determine the UMP size α test of $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_a : \sigma^2 > \sigma_0^2$. Derive an expression for the power function of this test in terms of the chi-squared distribution.