Exercises 4 STA 3000, 2020

1. Suppose $C_\gamma : [0,1] \times \mathcal{X} \to 2^\Psi$ is a γ-confidence region for $\psi = \Psi(\theta)$, namely,

$$\text{Prob}_\theta(\Psi(\theta) \in C_\gamma(u,x)) \geq \gamma$$

for every $\theta \in \Theta$ where $(u,x) \sim \text{Prob}_\theta$ given by $u \sim U(0,1)$ statistically independent of $x \sim P_\theta$. Now for each $\psi_0 \in \Psi$ define a test function $\varphi_{\psi_0} : \mathcal{X} \to [0,1]$ by

$$\varphi_{\psi_0}(x) = \text{Prob}(\psi_0 \notin C_\gamma(u,x) \,|\, x). \tag{1}$$

(a) Show that the power function of φ_{ψ_0} is given by

$$\beta(\theta) = E_\theta(\varphi_{\psi_0}) = \text{Prob}_\theta(\psi_0 \notin C_\gamma(u,x))$$

and so φ_{ψ_0} is size $1-\gamma$ for $H_0 = \{\psi_0\}$ vs $H_a = \{\psi_0\}^c$.
(b) Since $\{\varphi_{\psi_0} : \psi_0 \in \Psi\}$ is a set of $1-\gamma$ test functions for the hypothesis testing problems $H_0 = \{\psi_0\}$ vs $H_a = \{\psi_0\}^c$, the set function $C_\gamma^* : [0,1] \times \mathcal{X} \to 2^\Psi$ given by

$$C_\gamma^*(u,x) = \{\psi_0 : \varphi_{\psi_0}(x) < u\}$$

is, as shown in class, a γ-confidence region for $\psi = \Psi(\theta)$. Show that the coverage probabilities for C_γ^* are the same as those for C_γ.
(c) Prove that C_γ is a uniformly most accurate unbiased γ-confidence region for $\psi = \Psi(\theta)$ iff each φ_{ψ_0} defined in (1) is UMPU size $1-\gamma$ for $H_0 = \{\psi_0\}$ vs $H_a = \{\psi_0\}^c$ for each $\psi_0 \in \Psi$.
(d) Show that $C_\gamma^* = C_\gamma$ when C_γ is nonrandomized.
(e) Show that it is always possible to find a γ-confidence region with exact confidence γ.

2. Suppose that $x = (x_1, \ldots, x_n)$ are *i.i.d.* Bernoulli(θ).
(a) Determine the UMPU size α test for $H_0 = \{\theta_0\}$ versus $H_a = \{\theta_0\}^c$.
(b) Determine the uniformly most accurate unbiased $1-\alpha$ confidence interval for θ.

3. Suppose that $x = (x_1, \ldots, x_n)$ is a sample from an gamma(η, θ) distribution ($f_\theta(x) = \theta^\eta x^{\eta-1} \exp\{-\theta x\}/\Gamma(\eta)$ for $x > 0$) where η is known and $\theta > 0$ is unknown.
(a) Determine the UMP size α test of $H_0 : \theta \leq \theta_0$ versus $H_a : \theta > \theta_0$.
(b) Determine the UMPU size α test of $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$.

4. Suppose that $x = (x_1, \ldots, x_n)$ are *i.i.d.* $N(\mu, \sigma^2)$ where μ and σ^2 are completely unknown and we want to test $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_a : \sigma^2 \geq \sigma_0^2$.and consider the location group $G = (R, +)$ acting on R^n via $T_g x = x + g1$ where $1 \in R^n$ is the vector of ones. Obtain the optimal invariant size α test. Hint: find the optimal invariant test for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 \geq \sigma_0^2$ first.

5. Suppose that $x \sim N_2(0, \Sigma)$ statistically independent of $y \sim N_2(0, \Delta\Sigma)$ where $\Delta > 0$ and Σ p.d. are unknown.
(a) Let

$$G_1 = \left\{ \left[\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix}, c \right] \mid a_1, b, c > 0 \text{ and } a_2 \in R^1 \right\}$$

and define the product on G_1 by

$$\left[\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix}, c \right] \left[\begin{pmatrix} a_1' & a_2' \\ 0 & b \end{pmatrix}, c' \right] = \left[\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} \begin{pmatrix} a_1' & a_2' \\ 0 & b \end{pmatrix}, cc' \right].$$

Prove that this defines a group.
(b) For

$$g = \left[\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix}, c \right]$$

define the transformation $T_g : R^4 \to R^4$ by

$$T_g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} x \\ c \begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} y \end{pmatrix}.$$

Prove that this defines an action on R^4. Is this action free?
(c) Prove that the action defined in (b) leaves the model invariant and determine the induced action on Θ.
(d) Suppose $\Psi(\Sigma, \Delta) = \Delta$ and the loss function is given by $\rho(\psi/\Delta)$ where $\rho : R^1 \to [0, \infty)$ is convex and satisfies $\rho(1) = 0$. Prove that this decision problem is invariant under G_1 and determine the action on the Ψ space. Justify why we can restrict to nonrandomized rules.
(e) Prove that a nonrandomized rule d is equivariant iff $d(x, y) = k_1 y_2^2 / x_2^2$ for some k_1 and the optimal estimator is determined by finding the value k_1 that minimizes $E_f\left[\rho\left(k_1 y_2^2 / x_2^2\right)\right]$ where f is the bivariate standard normal density.
(f) Repeat (a)-(e) (modifying part (e) appropriately) but using the group

$$G_2 = \left\{ \left[\begin{pmatrix} a_1 & 0 \\ a_2 & b \end{pmatrix}, c \right] \mid a_1, b, c > 0 \text{ and } a_2 \in R^1 \right\}.$$

6. Suppose $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \overset{\text{i.i.d.}}{\sim} N_k(\theta, I)$ with $\theta \in \Theta = R^k$. Interest is in estimating $\Psi(\theta) = \theta$ and the loss function is $L(\theta, \psi) = ||\theta - \psi||^2$. Consider the group $G = R^k$ with $g_1 \cdot g_2 = g_1 + g_2$.
(a) Prove that G is a group.
(b) Suppose we define $T_g : R^{k \times n} \to R^{k \times n}$ by $T_g X = X + g\mathbf{1}'$. Prove that this defines an action of G on the sample space. Is the action free?
(c) Show that the group leaves the model invariant and determine the action of G on Θ and on Ψ. Show that G acts transitively on Ψ. Show that G leaves the decision function invariant.
(d) Show that

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i$$

is an equivariant mss. Explain why we can then base the search for an optimal equivariant estimator on $\bar{X}$.
(e) Show that the optimal equivariant estimator is $\bar{X}$.
(f) Suppose instead we want to estimate $\Psi(\theta) = ||\theta||^2$. Is this problem invariant under G.