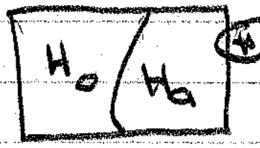
(6) Hypothesis Testing

- $\Theta = H_0 \cup H_a$, $H_0 \cap H_a = \phi$



- $\varphi(x)$ = prob. H_0 rejected given $x \in \mathcal{X}$
- $\varphi : \mathcal{X} \to [0,1]$
- $R(\theta,\varphi) = c_0 E_\theta(\varphi) I_{H_0}(\theta) + c_a(1-E_\theta(\varphi)) I_{H_a}(\theta)$
- no optimal φ so restrict to $\mathcal{D}_\alpha = \{\varphi : E_\theta(\varphi) \le \alpha \ \forall \theta \in H_0\}$ size α test fns
- set $c_0 = c_a = 1$ find $\varphi \in \mathcal{D}_\alpha$ that maximizes the power function $E_\theta(\varphi)$ for each $\theta \in H_a$

= $P_\theta(A) = \int_A f_\theta(x)\,\mu(dx)$ where μ = "volume measure"

Theorem ① (Neyman-Pearson, The Fundamental Lemma)
Suppose $\Theta = \{\theta_0, \theta_1\}$ and $H_0 = \{\theta_0\}$. Then

(i) $\exists\ \varphi \in \mathcal{D}_\alpha$ satisfying $E_\theta(\varphi) = \alpha$ and s.t. φ is of the form

$$\varphi(x) = \begin{cases} 1 & f_{\theta_1}(x)/f_{\theta_0}(x) > k_0 \\ \gamma & \quad " \quad = k_0 \\ 0 & \quad " \quad < k_0 \end{cases}$$

for some $k_0 \in \mathbb{R} \cup \{\infty\}$, $\gamma \in [0,1]$.

(ii) φ in (i) is MP size α.

(iii) If φ_* is MP size α then

$$\mu(\{x : \varphi_*(x) \neq \varphi(x),\ f_{\theta_1}(x)/f_{\theta_0}(x) \neq k_0\}) = 0$$

and $E_{\theta_0}(\varphi_*) = \alpha$ unless $E_{\theta_1}(\varphi_*) = 1$.

Corollary ② If φ is MP size α then $\alpha \le E_{\theta_1}(\varphi)$ with the inequality strict when $0 < \alpha < 1$.

Proof of the Fundamental Lemma (FL)

First note that $\{x : f_{\theta_0}(x) = f_{\theta_1}(x) = 0\}$ has P_{θ_0} and P_{θ_1} measure 0 and so can be deleted from $\mathfrak{X}$.

Proof of (i) and (ii)

Suppose $\alpha = 1$. Then putting $k_0 = 0$, $\gamma = 1$ we see that $\phi(s) \equiv 1$ satisfies (i) and $E_{\theta_0}(\phi) = 1 = E_{\theta_1}(\phi)$ and clearly ϕ is MP size α.

Suppose $\alpha = 0$. Then put $k_0 = \infty$, $\gamma = 1$ and define $\phi(x) = \begin{cases} 1 & f_{\theta_1}(x)/f_{\theta_0}(x) = \infty \\ 0 & < \infty \end{cases}$. Then $E_{\theta_0}(\phi) = 0$ and $E_{\theta_1}(\phi) = P_{\theta_1}(\{x : f_{\theta_0}(x) = 0\})$. If ϕ_* is of size 0 then $E_{\theta_0}(\phi_*) = 0$ implies $\phi_* = 0$ a.e. P_{θ_0}

so $\mu(\{x : a_y(x) > 0, f_{\theta_0}(x) > 0\}) = 0$.

Therefore $E_{\theta_1}(a_y) = \int_{\{x : a_y(x) > 0\}} a_y(x) f_{\theta_1}(x) \mu(dx)$

$= \int_{\{x : a_y(x) > 0, f_{\theta_0}(x) = 0\}} + \int_{\{x : a_y(x) > 0, f_{\theta_0}(x) > 0\}}$ $=0$

$= \int_{\{x : a_y(x) > 0, f_{\theta_0}(x) = 0\}} f_{\theta_1}(x) \mu(dx) \leq P_{\theta_1}(\{x : f_{\theta_0}(x) = 0\})$

$= E_{\theta_1}(a)$

<u>Suppose $0 < \alpha < 1$</u>

(i) Consider $1 - \alpha(k) = P_{\theta_0}(f_{\theta_1}(x)/f_{\theta_0}(x) \leq k)$.

Then $1 - \alpha(k)$ is the cdf of $f_{\theta_1}(x)/f_{\theta_0}(x)$ when $x \sim f_{\theta_0}$. Therefore $1 - \alpha(k)$ is an increasing function of k, $1 - \alpha(-\infty) = 0$, $1 - \alpha(\infty) = 1$ and it is right continuous. Let $k_0 = \inf\{k : 1 - \alpha \leq 1 - \alpha(k)\}$.

Then by right cty we have $1 - \alpha(k_0 - 0) \leq 1 - \alpha \leq 1 - \alpha(k_0)$. Further $P_{\theta_0}(f_{\theta_1}(x)/f_{\theta_0}(x) = k_0)$

$= (1 - \alpha(k_0)) - (1 - \alpha(k_0 - 0)) = \alpha(k_0 - 0) - \alpha(k_0)$.

Then define $a(x) = \begin{cases} 1 & f_{\theta_1}(x)/f_{\theta_0}(x) > k_0 \\ \dfrac{\alpha - \alpha(k_0)}{\alpha(k_0 - 0) - \alpha(k_0)} \text{ (0 when } \alpha(k_0-0) = \alpha(k_0)) & = \\ & < \end{cases}$

Then $E_{\theta_0}(\phi) = P_{\theta_0}\left(f_{\theta_1}(x)/f_{\theta_0}(x) > k_0\right)$

$+ \dfrac{\alpha-\alpha(k_0)}{\alpha(k_0-0)-\alpha(k_0)} P\left(f_{\theta_1}(x)/f_{\theta_0}(x) = k_0\right)$

$= \alpha(k_0) + \dfrac{\alpha-\alpha(k_0)}{\alpha(k_0-0)-\alpha(k_0)}\left(\alpha(k_0-0)-\alpha(k_0)\right) = \alpha.$

(ii) Consider ϕ_γ with ϕ as in (i). Then

$E_{\theta_1}(\phi) - E_{\theta_1}(\phi_\gamma) = \int_{\mathcal{X}} (\phi(x)-\phi_\gamma(x)) f_{\theta_1}(x)\,\mu(dx)$

$= \int_{\{x:\,\phi(x)-\phi_\gamma(x)\neq 0\}} (\phi(x)-\phi_\gamma(x)) f_{\theta_1}(x)\,\mu(dx)$

now $\{x : \phi(x)-\phi_\gamma(x) \neq 0\}$

$= \{x : \phi(x)-\phi_\gamma(x) < 0\} \cup \{x : \phi(x)-\phi_\gamma(x) > 0\}$

and $\{x : \phi(x)-\phi_\gamma(x) < 0\}$

$= \{x : f_{\theta_1}(x)/f_{\theta_0}(x) < k_0,\ \phi(x)-\phi_\gamma(x) < 0\}$

$\cup \{x : f_{\theta_1}(x)/f_{\theta_0}(x) = k_0,\ \phi(x)-\phi_\gamma(x) < 0\}$

since $f_{\theta_1}(x)/f_{\theta_0}(x) > k_0$ implies $\phi(x)-\phi_\gamma(x) = 1-\phi_\gamma(x) \geq 0$

$\geq \int_{\{x:\,f_{\theta_1}(x)/f_{\theta_0}(x) \leq k_0,\ \phi(x)-\phi_\gamma(x)<0\}} (\phi(x)-\phi_\gamma(x))\, k_0 f_{\theta_0}(x)\,\mu(dx) \ +$

$\int_{\{x:\,f_{\theta_1}(x)/f_{\theta_0}(x) \geq k_0,\ \phi(x)-\phi_\gamma(x)>0\}} (\phi(x)-\phi_\gamma(x))\, k_0 f_{\theta_0}(x)\,\mu(dx)$

$= k_0 \int_{\mathfrak{X}} (\phi(x) - \phi_*(x)) f_{\theta_0}(x)\, \mu(dx)$

$= k_0 (E_{\theta_0}(\phi) - E_{\theta_0}(\phi_*)) = k_0(\alpha - E_{\theta_0}(\phi_*)) \geq 0$

since ϕ_* is of size α. Therefore, we must have $E_{\theta_1}(\phi) \geq E_{\theta_1}(\phi_*)$ and ϕ is MP size α.

(iii) Let ϕ_* be a MP size α test and ϕ be as in (i) and $\mathfrak{X}_* = \{x : \phi(x) \neq \phi_*(x)\} \cap \{x : f_{\theta_1}(x) \neq k_0 f_{\theta_0}(x)\}$.

Then $0 \geq (E_{\theta_1}(\phi) - E_{\theta_1}(\phi_*)) - k_0(\alpha - E_{\theta_0}(\phi_*))$

$= \int_{\mathfrak{X}} (\phi(x) - \phi_*(x))(f_{\theta_1}(x) - k_0 f_{\theta_0}(x))\, \mu(dx)$

$= \int_{\mathfrak{X}_*} (\phi(x) - \phi_*(x))(f_{\theta_1}(x) - k_0 f_{\theta_0}(x))\, \mu(dx)$

≥ 0 since $(\phi(x) - \phi_*(x))(f_{\theta_1}(x) - k_0 f_{\theta_0}(x)) > 0$ on $\mathfrak{X}_*$

and so we must have that $\mu(\mathfrak{X}_*) = 0$. (This implies that $\phi_*(x) = \phi(x)$ when $f_{\theta_1}(x) \neq k_0 f_{\theta_0}(x)$ except perhaps on a set having μ-measure 0.)

If $\mu(\{x : f_{\theta_1}(x)/f_{\theta_0}(x) = k_0\}) = 0$ then

$\phi_* = \phi$ a.e. μ and $E_{\theta_1}(\phi_*) = \alpha$.

Consider the situation when $\mu(\{f_{\theta_1}(x)/f_{\theta_0}(x)=k_0\})>0$, and $E_{\theta_1}(\phi)=E_{\theta_1}(\phi_k)<1$. Suppose

$$\alpha > E_{\theta_0}(\phi_k) = \int_{\{f_{\theta_1}(x)/f_{\theta_0}(x)=k_0\}} \phi_k(x) f_{\theta_0}(x)\,\mu(dx)$$

$$+ \int_{\{f_{\theta_1}(x)/f_{\theta_0}(x)\neq k_0\}} \phi(x) f_{\theta_0}(x)\,\mu(dx)$$

which implies $\int_{\{\;=k_0\}} \phi_k(x) f_{\theta_0}(x)\,\nu(dx)$ $< \int_{\{\;=k_0\}} \phi(x) f_{\theta_0}(x)\,\mu(dx)$. Assume $k_0 \neq 0, \infty$

this implies $\cancel{\frac{1}{k_0}} \int_{\{\;=k_0\}} \phi_k(x) f_{\theta_1}(x)\,\mu(dx)$ $< \cancel{\frac{1}{k_0}} \int_{\{\;=k_0\}} \phi(x) f_{\theta_1}(x)\,\mu(dx)$ which implies

$E_{\theta_1}(\phi_k) < E_{\theta_1}(\phi)$ (x). When $k_0=0$ then $E_{\theta_1}(\phi)=1$. When $k_0=\infty$ then $E_{\theta_0}(\phi)=0$ then $\alpha=0$ since ϕ is exact size α and so $E_{\theta_0}(\phi_k)=0$.

Proof of Corollary (>)

Consider $\varphi_\alpha(x) \equiv \alpha$. Then $E_{\theta_1}(\varphi) \geq E_{\theta_1}(\varphi_\alpha) = \alpha$. Suppose $E_{\theta_1}(\varphi) = \alpha$. Then φ_α is MP size α which by (iii) implies $\mu(\{\{f_{\theta_1}(x) = k_0 f_{\theta_0}(x)\}\}) = 1$ which implies $k_0 = 1$ since f_θ is a density for $\theta \in \Theta$. But this implies $f_{\theta_0} = f_{\theta_1}$ (x).

- note - if $\mu(\{f_{\theta_1}(x) = k f_{\theta_0}(x)\}) = 0$ $\forall k$ then randomized tests are not required and MP size α test is unique

- to find k_0 and γ calculate

$\alpha(k) = P_{\theta_0}(f_{\theta_1}(x)/f_{\theta_0}(x) > k)$ and

find k_0 s.t. $\alpha(k_0 - \varepsilon) > \alpha$ $\forall \varepsilon > 0$ and $\alpha(k_0) \leq \alpha$ and then put

$$\gamma = \begin{cases} \dfrac{\alpha - \alpha(k_0)}{\alpha(k_0 - 0) - \alpha(k_0)} & \text{when } \alpha(k_0) < \alpha \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

eg ① $x = (x_1, \ldots, x_n) \overset{iid}{\sim} N(\mu, 1)$

- $\mu \in \{\mu_0, \mu_1\}$ with $\mu_0 < \mu_1$ and test $H_0 = \{\mu_0\}$ versus $H_a = \{\mu_1\}$

- we can restrict to tests fns that depend on the data only thru a sufficient statistic and $\bar{x}$ is sufficient

- $\bar{x} \overset{H_0}{\sim} N(\mu_0, 1/n)$, $\bar{x} \overset{H_a}{\sim} N(\mu_1, 1/n)$

- to find k_0 and γ

$f_1(\bar{x})/f_0(\bar{x}) > k_0$

iff $\log(f_1(\bar{x})/f_0(\bar{x})) = -\frac{n}{2}(\bar{x}-\mu_1)^2 + \frac{n}{2}(\bar{x}-\mu_0)^2 > \log k_0$

iff $-(\bar{x}-\mu_1)^2 + (\bar{x}-\mu_0)^2 > \frac{2}{n}\log k_0$

iff $2(\mu_1-\mu_0)\bar{x} > \frac{2}{n}\log k_0 + \mu_1^2 - \mu_0^2$

iff $\bar{x} > \frac{1}{2(\mu_1-\mu_0)}\left(\frac{2}{n}\log k_0 + \mu_1^2 - \mu_0^2\right)$ (note $\mu_1 > \mu_0$)

iff $\sqrt{n}(\bar{x}-\mu_0) > \frac{\sqrt{n}}{2(\mu_1-\mu_0)}\left(\frac{2}{n}\log k_0 + \mu_1^2 - \mu_0^2\right) - \sqrt{n}\mu_0 = k_0'$

$\therefore P_{\mu_0}(f_1(\bar{x})/f_0(\bar{x}) > k_0) = P_{\mu_0}(\sqrt{n}(\bar{x}-\mu_0) > k_0')$

$= 1 - \Phi(k_0')$ and we choose $k_0' = z_{1-\alpha}$ where

$z_{1-\alpha}$ is the α-th quantile of the $N(0,1)$ dist'n and for this choice of k_0' (which determines k_0) we have $P_{\mu_0}(f_1(\bar{x})/f_0(\bar{x}) > k_0) = \alpha$ and so $\gamma = 0$. Therefore the MP size α test for testing $\{\mu_0\}$ versus $\{\mu_1\}$ when $\mu_1 > \mu_0$ is $\varphi(x) = 1$ when $\sqrt{n}(\bar{x} - \mu_0) > z_{1-\alpha}$ and $\varphi(x) = 0$ otherwise (so we reject H_0 whenever $\bar{x}$ is large).

$$\boxed{\begin{array}{l} \underline{\text{note}}\ \alpha = .05,\ \mu_0 = 0,\ \mu_1 = 1,\ z_{.95} = 1.644854 \\ \log k_0 = \sqrt{n}\,(1.644854) - n/2 > 0 \text{ iff } n \le 11 \\ \text{so can reject } H_0 \text{ even when evidence favors } \mu_0 \end{array}}$$

Notice that this does not depend on μ_1 and so φ is MP size α for $H_0 = \{\mu_0\}$ versus $H_a = (\mu_0, \infty)$ (i.e. for any alternative $> \mu_0$). Also if $\mu < \mu_0$ then by the same argument φ is the MP size $E_\mu(\varphi)$ test of $H_0 = \{\mu\}$ vs $H_a = \{\mu_0\}$ and by the Corollary $E_\mu(\varphi) \le E_{\mu_0}(\varphi) = \alpha$. Therefore, φ is size α for $H_0 = (-\infty, \mu_0]$ versus $H_a = (\mu_0, \infty)$. Now suppose that

φ_y is size α for $H_0 = (-\infty, \mu_0]$ vs $H_a = (\mu_0, \infty)$.
Then φ_y is size α for $H_0 = \{\mu_0\}$ vs $H_a = \{\mu_1\}$
for $\mu_1 > \mu_0$ and so $E_{\mu_1}(\varphi_y) \le E_{\mu_1}(\varphi)$.
This proves that φ is UMP size α
for $H_0 = (-\infty, \mu_0]$ vs $H_a = (\mu_0, \infty)$.

- <u>**note**</u> - we can repeat this argument to
find the UMP size α test for
$H_0 = [\mu_0, \infty)$ vs $H_a = (-\infty, \mu_0)$ and obtain
$\varphi_y(x) = 1$ when $\sqrt{n}(\bar{x} - \mu_0) < z_\alpha$.
- now consider the two-sided problem
$H_0 = \{\mu_0\}$ vs $H_a = (-\infty, \mu_0) \cup (\mu_0, \infty)$
and suppose φ_{**} is UMP size α for this.
- but then φ_{**} is size α for $H_0 = \{\mu_0\}$
vs $H_0 = \{\mu_1\}$ and is MP, so by FL(iii)
we have $\varphi_{**}(x) = \varphi(x)$ (when $\mu_1 > \mu_0$) and

$\phi_{x}(x) = \phi_{y}(x)$ (when $\mu_1 < \mu_0$) which is is impossible

- therefore there is no UMP size α test for the two-sided problem (have to restrict to unbiased tests and then $\phi(x) = 1$ when $\sqrt{n}|\bar{x}-\mu_0| > z_{\alpha/2}$ and 0 otherwise, can be shown to be UMP unbiased of size α).

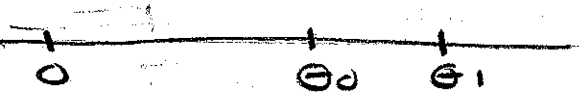
eg (2) – $\underline{x} = (x_1, \dots, x_n) \overset{iid}{\sim} U(0, \theta)$

– $H_0 = \{\theta_0\}$ vs $H_a = \{\theta_1\}$ where $\theta_0 < \theta_1$

– we can restrict to tests which are fns of the suff. stat. $x_{(n)}$

– $P_\theta(X_{(n)} \le x) = P_\theta(X_1 \le x, \dots, X_n \le x)$
$= \prod_{i=1}^{n} P_\theta(X_i \le x) = \left(\frac{1}{\theta}\int_0^x dz\right) = \frac{x^n}{\theta^n}$
and so $f_\theta(u) = n u^{n-1}/\theta^n$

– now



$$f_{\theta_1}(x_{(n)})/f_{\theta_0}(x_{(n)}) = \begin{cases} \left(\frac{\theta_0}{\theta_1}\right)^n & 0 < x_{(n)} < \theta_0 \\ \infty & \theta_0 < x_{(n)} < \theta_1 \end{cases}$$

– so $P_{\theta_0}\left(f_{\theta_1}(x_{(n)})/f_{\theta_0}(x_{(n)}) > k_0\right) = \begin{cases} 1 & k_0 < \left(\frac{\theta_0}{\theta_1}\right)^n \\ 0 & k_0 > \left(\frac{\theta_0}{\theta_1}\right)^n \end{cases}$

– $\therefore$ for $0 < \alpha < 1$, $k_0 = \left(\frac{\theta_0}{\theta_1}\right)^n$ and $\gamma = \alpha$

and $\varphi(x) = \begin{cases} \alpha & 0 < x_{(n)} \le \theta_0 \\ 1 & x_{(n)} > \theta_0 \end{cases}$

– so the MP size α test here is to randomly reject with probability α whenever $0 \le x_{(n)} \le \theta_0$ and reject when $x_{(n)} > \theta_0$

- the "boundary" here is the set $(0, \theta_0]$ (this is continuous model where we have to randomize)

- also this test does not depend on θ_1 so it is UMP size α for $H_0 = \{\theta_0\}$ vs $H_a = (\theta_0, \infty)$

- also for $\theta < \theta_0$ we have: $E_\theta(\varphi) \leq \alpha$ and so (as in the previous example) this test is UMP size α for $H_0 = (0, \theta_0]$ vs $H_a = (\theta_0, \infty)$.

eg ③ $\underline{x} = (x_1, \ldots, x_n) \overset{iid}{\sim}$ Poisson (λ)

- $H_0 = \{\lambda_0\}$ vs $H_a = \{\lambda_1\}$ $\quad \lambda_0 < \lambda_1$

- then $n\bar{x}$ is sufficient so we can base the test on $n\bar{x} \sim$ Poisson (λ)

- $f_{\lambda_1}(n\bar{x}) / f_{\lambda_0}(n\bar{x}) = \left(\frac{\lambda_1}{\lambda_0}\right)^{n\bar{x}} e^{-n(\lambda_1 - \lambda_0)} > k$

  iff $\quad n\bar{x} \log(\lambda_1/\lambda_0) - n(\lambda_1 - \lambda_0) > \log k$

  iff $\quad n\bar{x} > (\log(\lambda_1/\lambda_0))^{-1} (k + n(\lambda_1 - \lambda_0))$

  (note $\log(\lambda_1/\lambda_0) > 0$)

- so let $k_0 \in \mathbb{N}$ be s.t. $P_{\lambda_0}(n\bar{x} > k_0) \le \alpha$ and $P_{\lambda_0}(n\bar{x} < k_0 - 1) > \alpha$

- then MP size α test of $H_0 = \{\lambda_0\}$ vs $H_a = \{\lambda_1\}$ is

$$\varphi(x) = \begin{cases} 1 & n\bar{x} > k_0 \\ \dfrac{\alpha - P_{\lambda_0}(n\bar{x} > k_0)}{P_{\lambda_0}(n\bar{x} > k_0 - 1) - P_{\lambda_0}(n\bar{x} > k_0)} = \gamma & = \\ 0 & < \end{cases}$$

- since this test does not involve λ_1 it is MP size α for $H_0 = \{\lambda_0\}$ vs $H_a = (\lambda_0, \infty)$

- now suppose $\lambda < \lambda_0$ and consider the problem $H_0 = \{\lambda\}$ vs $H_a = \{\lambda_0\}$

- Then the same argument shows that φ is MP of size $E_\lambda(\varphi)$ for this problem and so by the Corollary of the FL $E_\lambda(\varphi) \le E_{\lambda_0}(\varphi) = \alpha$

- Therefore φ is of size α for $H_0 = (0, \lambda_0]$ vs $H_a = (\lambda_0, \infty)$ and so must be UMP size α for this problem