(b) Monotone Likelihood Ratio

- assume $\{f_\theta : \theta \in \Theta\}$ is s.t. $\Theta \subseteq \mathbb{R}^1$ and MSS $T(x) \in \mathbb{R}$, so suppose $x \in \mathbb{R}^1$

- $\{f_\theta : \theta \in \Theta\}$ has monotone likelihood ratio (MLR) form if whenever $\theta_1 < \theta_2$, the $f_{\theta_2}(x) / f_{\theta_1}(x)$ is a monotone (increasing or decreasing) fn of x always in the same direction.

Theorem ③ If $\{f_\theta : \theta \in \Theta\}$ has strictly increasing MLR form, then a UMP size α test exists for $H_0 : \theta \leq \theta_0$ vs $H_a : \theta > \theta_0$ and is of the form

$$\varphi(x) = \begin{cases} 1 & x > x_0 \\ \gamma & = \\ 0 & < \end{cases}$$

where x_0, γ are determined to give exact size α.

Proof: Let $\theta_0 < \theta_1$. Then by the FL the MP size α test of θ_0 vs θ_1 is of the form

$$\varphi_\gamma(x) = \begin{cases} 1 & f_{\theta_1}(x)/f_{\theta_0}(x) > k_0 \\ \gamma & = k_0 \\ 0 & < k_0 \end{cases}$$

Since the model is of strict increasing MLR form $\exists$ at most one x_0 s.t. $f_{\theta_1}(x_0)/f_{\theta_0}(x_0) = k_0$ and so $\varphi_\gamma = \varphi$.

Since it is independent θ_1, ϕ is also MP size α for θ_0 vs θ_1 for $\theta_1 > \theta_0$ and thus UMP size α for $\theta = \theta_0$ vs $\theta > \theta_0$.

For any $\theta_0' < \theta_0$ we have that ϕ is MP size $E_{\theta_0'}(\phi)$ for θ_0' vs θ_0 and so $E_{\theta_0'}(\phi) \leq E_{\theta_0}(\phi) = \alpha$ by the Corollary to the FL so ϕ is size α $H_0: \theta \leq \theta_0$ vs $H_a: \theta > \theta_0$.
If ϕ' is size α for then it is size α for $H_0: \theta = \theta_0$ vs $H_a: \theta > \theta_0$ so $E_\theta(\phi') \leq E_\theta(\phi)$ $\forall \theta > \theta_0$ so ϕ is UMP size α for $H_0: \theta \leq \theta_0$ vs $H_a: \theta > \theta_0$.

<u>Corollary</u> ④ $E_\theta(\phi)$ is monotone increasing.
Proof: We have shown that for any $\theta_1 < \theta_2$ then ϕ is MP size $E_{\theta_1}(\phi)$ for θ_1 vs θ_2 and so by the Corollary to the FL $E_{\theta_1}(\phi) \leq E_{\theta_2}(\phi)$

<u>Corollary</u> ⑤ For $\theta \leq \theta_0$, $E_\theta(\phi)$ is minimized among all size α tests for $\theta \leq \theta_0$ vs $\theta > \theta_0$

note: can weaken to just increasing or decreasing but then boundary becomes an interval.

Proof: Reverse the FL to min $E_\theta(\varphi)$ subject to $E_{\theta_0}(\varphi) \geq \alpha$ and note that this will just reverse the roles of 1 and 0 in definition of φ.

- **note** - if $\{f_\theta : \theta \in \Theta\}$ is MLR strictly decreasing then the UMP size α for $\theta \leq \theta_0$ vs $\theta > \theta_0$ is of the form

$$\varphi(x) = \begin{cases} 1 & x < x_0 \\ \gamma & = \\ 0 & > \end{cases}$$

- similar results for $H_0: \theta \geq \theta_0$ vs $H_a: \theta < \theta_0$.

eg 1-parameter exponential family

- $f_\theta(x) = \exp\{\theta t(x) - A(\theta)\}$

so $\dfrac{f_{\theta_2,T}(t)}{f_{\theta_1,T}(t)} = \exp\{(\theta_2 - \theta_1)t - (A(\theta_2) - A(\theta_1))\}$

$\therefore$ model for MSS, $t(x)$ is of MLR form (increasing)

- Poisson(λ), Binomial(n,p), $N(\mu,1)$

(c) Generalized Fundamental Lemma

Theorem 6 (Generalized Fundamental Lemma)

Let $f_0, f_1, \ldots, f_m$ be μ-integrable on $\mathcal{X}$. For fixed positive $k_1, \ldots, k_m$ and $\gamma \in [0,1]$ define

$$\phi_0(x) = \begin{cases} 1, & f_0(x) > \sum_{i=1}^m k_i f_i(x) \\ \gamma & = \\ 0 & < \end{cases}$$

Then ϕ_0 maximizes $\int_{\mathcal{X}} \phi(x) f_0(x) \mu(dx)$ among all test fns satisfying

$$\int_{\mathcal{X}} \phi(x) f_i(x) \mu(dx) \le \int_{\mathcal{X}} \phi_0(x) f_i(x) \mu(dx)$$

for $i = 1, \ldots, m$.

- note Neyman-Pearson $m=1$, $f_0 = f_{\theta_1}$, $f_1 = f_{\theta_0}$

Corollary 7 If $k_1, \ldots, k_m, \gamma$ are chosen s.t.

$\int_{\mathcal{X}} \phi_0(x) f_i(x) \mu(dx) = \alpha$ for $i = 1, \ldots, m$ and $f_0, \ldots, f_m$ are pdfs then $\int_{\mathcal{X}} \phi_0(x) f_0(x) \mu(dx) \ge \alpha$.

- note the f_i need not be pdf's

Proof of Theorem (3)

Let $a : \mathcal{X} \to [0,1]$ satisfy $\int_{\mathcal{X}} a(x) f_i(x) \mu(dx)$

$\le \int_{\mathcal{X}} a_0(x) f_i(x) \mu(dx)$ for $i = 1, \ldots, m$ and

suppose $\int_{\mathcal{X}} a(x) f_0(x) \mu(dx) \geq \int_{\mathcal{X}} a_0(x) f_0(x) \mu(dx)$,

Then $0 \le \int_{\mathcal{X}} (a(x) - a_0(x)) f_0(x) \mu(dx)$

$= \int_{\{a(x) - a_0(x) \neq 0\}} (a(x) - a_0(x)) f_0(x) \mu_0(dx)$

$= \int_{\{f_0(x) \geq \sum k_i f_i(x),\ a(x) - a_0(x) < 0\}}$

$\qquad + \int_{\{f_0(x) \le \sum k_i f_i(x),\ a(x) - a_0(x) > 0\}}$

$\le \int_{\{\ \}} (a(x) - a(x_0)) \sum k_i f_i(x) \mu(dx)$

$\qquad + \int_{\{\ \}} (a(x) - a(x_0)) \sum k_i f_i(x) \mu(dx)$

$= \sum k_i \int_{\mathcal{X}} (a(x) - a(x_0)) f_i(x) \mu(dx)$

$= \sum k_i \int_{\mathcal{X}} a(x) f_i(x) \mu(dx) - \sum k_i \int_{\mathcal{X}} a_0(x) f_i(x) \mu(dx)$

≤ 0 since a satisfies the inequalities and so

$\int_{\mathcal{X}} a(x) f_0(x) \mu(dx) = \int_{\mathcal{X}} a_0(x) f_0(x) \mu(dx)$.

Proof of Corollary ⑦ The fn $\varphi_\alpha(x) \equiv \alpha$ satisfies $\int_{\mathcal{X}} \varphi_\alpha(x) f_i(x) \mu(dx) = \alpha$ $i = 0, \ldots, m$ and the result follows.

eg <u>locally most powerful tests</u>

- suppose we want to test $H_0: \theta \le \theta_0$ vs $H_a: \theta > \theta_0$ and we don't have MLR so it is unlikely that a UMP test exists

- relax to finding a test good near θ_0

- $$\frac{\partial}{\partial \theta} E_\theta(\varphi) \overset{\text{under conditions}}{=} \int_{\mathcal{X}} \varphi(x) \frac{\partial f_\theta(x)}{d\theta} \mu(dx)$$
$$= E_\theta(\varphi(x) S(\theta|x))$$

- so we want φ s.t. $E_{\theta_0}(\varphi) = \alpha$ and

$E_{\theta_0}(\varphi(x) S(\theta|x))$ is maximized

(maximizing the slope of the power fn at θ_0)

- called a <u>locally most powerful size α test</u> (when it exists)

- put $f_0 = S(\theta_0|\cdot) f_{\theta_0}$, $f_1 = f_{\theta_0}$ in GFL and look for size α test of the form

$$\varphi_0(x) = \begin{cases} 1 & S(\theta_0|x) > k_0 \\ \gamma & \quad = \\ 0 & \quad < k_0 \end{cases}$$

eg <u>location Cauchy</u>

- $x = (x_1, \ldots, x_n) \overset{iid}{\sim} \dfrac{1}{\pi(1+(x-\theta)^2)}$ $\theta \in \mathbb{R}$.
- test $H_0: \theta = \theta_0$ vs $H_a: \theta = \theta_1$ where $\theta_1 > \theta_0$
- FL says MP size α is of the form

$$\varphi(x) = \frac{\prod_{i=1}^n (1+(x_i-\theta_0)^2)}{\prod_{i=1}^n (1+(x_i-\theta_1)^2)} > k_0$$

for some k_0 (simulate under θ_0 to find k_0)

- but dependent on θ_1 so not UMP
- $l(\theta|x) = -n\log\pi - \sum_{i=1}^n \log(1+(x_i-\theta)^2)$

$$S(\theta|x) = 2\sum_{i=1}^n \frac{x_i-\theta}{1+(x_i-\theta)^2}$$

- find k_0 st $\alpha = P_{\theta_0}\left(\sum_{i=1}^n \dfrac{x_i-\theta_0}{1+(x_i-\theta_0)^2} \geq k_0\right)$

$$= P_0\left(\sum_{i=1}^n \frac{x_i}{1+x_i^2} \geq k_0\right)$$

- simulation when $n = 20$, $k_0 = 2.55$