7 Confidence Regions

- based on $\{f_\theta : \theta \in \Theta\}$ and $\psi = \Psi(\theta)$ we want $C_\gamma : \mathcal{X} \to 2^{\Psi}$ s.t. C_γ is a <u>γ-confidence</u> region for ψ, namely,

$$P_\theta\left(\Psi(\theta) \in C_\gamma(x) \right) \geq \gamma \quad \forall \theta \in \Theta$$

- this is not a decision problem (what is the correct action?) but it can be related to a decision problem

- for $\psi_0 \in \Psi$ let ϕ_{ψ_0} be a size $(1-\gamma)$ test fn for $H_0 = \{\psi_0\}$ vs $H_a = \{\psi_0\}^c$

- let $u \in [0,1]$ and define $C : [0,1] \times \mathcal{X} \to 2^{\Psi}$ by

$$C_\gamma(u, x) = \{ \psi_0 : \phi_{\psi_0}(x) < u \}$$

$\phi_{\psi_0}(x) = 1$ then $\psi_0 \notin C_\gamma(u,x)$
$\phi_{\psi_0}(x) = 0$ then $\psi_0 \in C_\gamma(u,x)$
$\phi_{\psi_0}(x) = \gamma$ then $\psi_0 \in C_\gamma(u,x)$ whenever $\gamma < u$

= set of ψ_0 which would be accepted when we observe x and we reject when $u \leq \phi_{\psi_0}(x)$

<u>**Lemma 1**</u> If $U \sim U(0,1)$ ind of $x \sim P_\theta$ then
$$\mathrm{Prob}\left(\Psi(\theta) \notin C_\gamma(U, x) \mid x \right) = \phi_{\Psi(\theta)}(x)$$

Proof: $\mathrm{Prob}\left(\Psi(\theta) \notin C_\gamma(U, x) \mid x \right)$

$$= \mathrm{Prob}\left(U \leq \phi_{\Psi(\theta)}(x) \mid x \right) = \phi_{\Psi(\theta)}(x)$$

- if ϕ_{ψ_0} is nonrandom, namely, $\phi_{\psi_0}(x) = 0$ or 1 for each x, then for $u \in (0,1)$ (ignore $u=0,1$)

$$C_\gamma(u,x) \equiv \{\psi_0 : \phi_{\psi_0}(x) < u\} = \{\psi_0 : \phi_{\psi_0}(x) = 0\}$$

$$= C_\gamma(x) \quad \text{as we can drop dependence on } u$$

Lemma ② $\text{Prob}(\Psi(\theta^*) \in C_\gamma(u,x) \mid \theta) = 1 - E_\theta(\phi_{\Psi(\theta^*)})$

Proof: $\text{Prob}(\Psi(\theta^*) \in C_\gamma(u,x) \mid \theta)$

$= \text{Prob}(\phi_{\Psi(\theta^*)}(x) < u \mid \theta) = 1 - \text{Prob}(\phi_{\Psi(\theta^*)}(x) \geq u \mid \theta)$

$\overset{\text{Lemma ①}}{=} 1 - E_\theta(\phi_{\Psi(\theta^*)})$.

Corollary ③ $\text{Prob}(\Psi(\theta) \in C_\gamma(u,x) \mid \theta) \geq \gamma$

Proof: Immediate since $E_\theta(\phi_{\Psi(\theta)}) \leq 1-\gamma \ \forall \theta$.

- so $C_\gamma(u,x)$ is a "randomized" confidence region for $\psi = \Psi(\theta)$: operationally, after observing x, generate $u \sim U(0,1)$ and put ψ_0 in $C(u,x)$ whenever $\phi_{\psi_0}(x) < u$.

Def A γ-confidence region C_γ for $\psi = \Psi(\theta)$ is <u>uniformly most accurate</u> if $\forall \theta, \theta^* \in \Theta$ with $\Psi(\theta^*) \neq \Psi(\theta)$ we have that $\text{Prob}(\Psi(\theta^*) \in C_\gamma(u,x) \mid \theta)$ is minimized among all γ-confidence regions.

<u>Lemma</u> 4 If ϕ_{ψ_0} is UMP size $(1-\gamma)$ for $H_0 = \{\psi_0\}$ versus $H_a = \{\psi_0\}^c$ for each $\psi_0 \in \Psi$ then the corresponding γ-confidence region is UMA.

Proof: We have that, from Lemma 2

$$1 - E_\theta(\phi_{\Psi(\theta_*)}) = \text{Prob}(\Psi(\theta_*) \in C_\gamma(U,X) \mid \theta)$$

and maximizing $E_\theta(\phi_{\Psi(\theta_*)})$ when $\Psi(\theta) \neq \Psi(\theta_*)$ is equivalent to minimizing $\text{Prob}(\Psi(\theta_*) \in C_\gamma(U,X) \mid \theta)$ when $\Psi(\theta_*) \neq \Psi(\theta)$.

<u>Def</u> A γ-confidence region C_γ is <u>unbiased</u> if $\text{Prob}(\Psi(\theta_*) \in C_\gamma(U,X) \mid \theta) \le \text{Prob}(\Psi(\theta) \in C_\gamma(U,X) \mid \theta)$ $\forall\, \theta, \theta_* \in \Theta$.

<u>Lemma</u> 5 A γ-confidence region C_γ is unbiased iff the corresponding tests are unbiased.

Proof: We have that if $\Psi(\theta_*) \neq \Psi(\theta)$ then

$$\text{Prob}(\Psi(\theta_*) \in C_\gamma(U,X) \mid \theta) = 1 - E_\theta(\phi_{\Psi(\theta_*)})$$

and so C_γ is unbiased iff $1 - E_\theta(\phi_{\Psi(\theta_*)}) \le 1 - E_\theta(\phi_{\Psi(\theta)})$ or iff $\phi_{\Psi(\theta)}$ is unbiased.

$\gamma = 1-\alpha$

$\alpha = 1-\gamma \quad 1-\alpha/2 = 1-\frac{1-\gamma}{2} = \frac{1+\gamma}{2}$

71

eg <u>location-normal</u>

- $\phi_{\mu_0}(\bar{x}) = \begin{cases} 1 & \sqrt{n}|\bar{x}-\mu_0| > z_{\frac{1+\gamma}{2}} \\ 0 & \text{otherwise} \end{cases}$ nonrandomized so corresponding conf. region is nonrandomized

is the UMPU size $1-\gamma$ test

- $$\begin{aligned} C_\gamma(\bar{x}) &= \{\mu_0 : \phi_{\mu_0}(\bar{x}) = 0\} \\ &= \{\mu_0 : \sqrt{n}|\bar{x}-\mu_0| \le z_{\frac{1+\gamma}{2}}\} \\ &= \{\mu_0 : -z_{\frac{1+\gamma}{2}} \le \sqrt{n}(\bar{x}-\mu_0) \le z_{\frac{1+\gamma}{2}}\} \\ &= \left[\bar{x} - \frac{1}{\sqrt{n}} z_{\frac{1+\gamma}{2}},\ \bar{x} + \frac{1}{\sqrt{n}} z_{\frac{1+\gamma}{2}}\right] \end{aligned}$$

is UMAU γ-confidence interval for μ.
and note it is also a likelihood interval

Lemma 6 If $C_\gamma : [0,1] \times \mathcal{X} \to 2^{\Psi}$ is a γ-confidence region for $\psi = \Psi(\theta)$, then

$$\alpha_{\Psi(\theta)}(x) = \text{Prob}\,(\Psi(\theta) \notin C_\gamma(u,x) \mid x)$$

is a size $(1-\gamma)$ test fn for $H_0 = \{\psi_0\}$ vs $H_a = \{\psi_0\}^c$.

Proof: Clearly $\alpha_{\Psi(\theta)} : \mathcal{X} \to [0,1]$ and

$$E_\theta(\alpha_{\Psi(\theta)}) = \text{Prob}\,(\Psi(\theta) \notin C_\gamma(u,x) \mid \theta)$$
$$= 1 - \text{Prob}\,(\Psi(\theta) \in C_\gamma(u,x) \mid \theta) \le 1-\gamma.$$

- note for α_ψ in Lemma 6

$$\{\psi_0 : \alpha_{\psi_0}(x) < u\} = \{\Psi(\theta) : \alpha_{\Psi(\theta)} < u\}$$
$$= \{\Psi(\theta) : \text{Prob}\,(\Psi(\theta) \notin C_\gamma(u,x) \mid x) < u)\}$$

- then $\text{Prob}\,(\Psi(\theta_*) \notin \{\psi_0 : \alpha_{\psi_0}(x) < u\} \mid x)$

$$= \text{Prob}\,(\text{Prob}\,(\Psi(\theta_*) \notin C_\gamma(u,x) \mid x) \ge u \mid x)$$
$$= \text{Prob}\,(\Psi(\theta_*) \notin C_\gamma(u,x) \mid x)$$

- so these probs are the same but it is not necessarily true that $C_\gamma(u,x) = \{\psi_0 : \alpha_{\psi_0}(x) < u\}$ except in the nonrandomized case.

- another approach to choosing a γ-CR is to choose the one minimizing

$$E(\nu(C_\gamma(u,x)) \mid \theta) \quad \forall \theta$$

where $\nu(A)$ is a measure of size of set A (e.g. volume)

Lemma 7 $E(\nu(C_\gamma(u,x)) \mid \theta)$

$$= \int_{\Psi \setminus \{\Psi(\theta)\}} \text{Prob}(\psi \in C_\gamma(u,x) \mid x, \theta)\, \nu(d\psi) + \nu(\{\Psi(\theta)\})\, \text{Prob}(\Psi(\theta) \in C_\gamma(u,x) \mid \theta)$$

Proof: $E(\nu(C_\gamma(u,x)) \mid \theta)$

$$= \int_{\mathcal{X}} \int_0^1 \int_{\Psi} I_{C_\gamma(u,x)}(\psi)\, \nu(d\psi)\, du\, P_\theta(dx)$$

$$= \int_{\Psi} \int_{\mathcal{X}} \int_0^1 I_{C_\gamma(u,x)}(\psi)\, du\, P_\theta(dx)\, \nu(d\psi)$$

$$= \int_{\Psi} \text{Prob}(\psi \in C_\gamma(u,x) \mid \theta)\, \nu(d\psi)$$

$$= \int_{\Psi \setminus \{\Psi(\theta)\}} \quad + \nu(\{\Psi(\theta)\})\, \text{Prob}(\Psi(\theta) \in C_\gamma(u,x) \mid \theta)$$

Corollary 8 If $\nu(\{\psi\})$ is constant (= 0) then a UMA γ-CR for $\Psi(\theta)$ minimizes $E(\nu(C_\gamma(u,x)) \mid \theta)$, and conversely when an exact UMA γ-CR exists.

Corollary 9 If $\nu(\{\psi\})$ is constant then a exact γ-CR that minimizes $E(\nu(C_\gamma(u,x)) \mid \theta)$ minimizes the ν-average of the prob of covering a false value.

- the implication of this is that we can't avoid randomized CI's if our criteria are exact size and minimizing expected volume.

eg **Bernoulli(θ)**

Brown, Cai and Das Gupta (2000) AS, 30, 1, 160-201 — compare intervals for θ (i) coverage prob (ii) expected length

- optimal CI for θ is a randomized interval which is exact size γ and UMAU and so minimizes expected length

- but nobody uses it because randomized

- what are the other proposals