Unbiasedness II

(c) Hypothesis Testing

- H_0 vs H_a, $\psi = \begin{cases} 0 & \text{accept } H_0 \\ 1 & \text{accept } H_a \end{cases}$

$$L(\theta, \psi) = c_0 I_{\{1\}}(\psi) I_{H_0}(\theta) + c_a I_{\{0\}}(\psi) I_{H_a}(\theta)$$

- for unbiased $\varphi : \mathcal{X} \to [0,1]$, $\varphi(x) = \delta(x, \{1\})$

$$R(\theta, \varphi) = c_0 E_\theta(\varphi) I_{H_0}(\theta) + c_a(1 - E_\theta(\varphi)) I_{H_a}(\theta)$$

$$\leq \int_{\mathcal{X}} \int_{\{0,1\}} L(\theta_*, \psi)\, \delta(x, d\psi)\, P_\theta(dx)$$

$$= c_0 E_\theta(\varphi) I_{H_0}(\theta_*) + c_a (1 - E_\theta(\varphi)) I_{H_a}(\theta_*)$$

iff $\begin{cases} c_0 E_\theta(\varphi) \leq c_a(1 - E_\theta(\varphi)) & \theta \in H_0,\ \theta_* \in H_a \\ c_a(1 - E_\theta(\varphi)) \leq c_0 E_\theta(\varphi) & \theta \in H_a,\ \theta_* \in H_0 \end{cases}$

iff $\begin{cases} E_\theta(\varphi) \leq c_a/(c_0 + c_a) & \theta \in H_0 \\ E_\theta(\varphi) \geq c_a/(c_0 + c_a) & \theta \in H_a \end{cases}$

- taking c_0, c_a s.t. $\alpha = c_a/(c_0 + c_a)$ we have that a size α φ is unbiased iff $E_\theta(\varphi) \leq \alpha$ $\forall \theta \in H_0$, $E_\theta(\varphi) \geq \alpha$ $\forall \theta \in H_a$ (prob. of rejecting H_0 when it is true is always no greater than rejecting H_0 when it is false)

- note $\phi_\alpha(x) \equiv \alpha$ is unbiased so in any class of tests that includes ϕ_α, an "optimal" test is unbiased.

+ similarity and Neyman structure.

Def A test ϕ is similar on $\Theta_0 \subseteq \Theta$ if $E_\theta(\phi)$ is constant on Θ_0.

- $\Theta = H_0 \cup H_a \quad H_0 \cap H_a = \phi$ and with a topology on Θ let ∂H_0 = boundary of H_0

Lemma ① If ϕ is unbiased size α and $E_\theta(\phi)$ is cont. at each $\theta \in \partial H_0$ then ϕ is similar of exact size α on ∂H_0.

Proof: Let $\{\theta_n\} \subseteq H_0$, $\{\theta_n'\} \subseteq H_a$ with

$\theta_n \to \theta$, $\theta_n' \to \theta$ for a $\theta \in \partial H_0$ then

$\lim E_{\theta_n}(\phi) = E_\theta(\phi) \leq \alpha \leq \lim E_{\theta_n'}(\phi) = E_\theta(\phi)$

and so $E_\theta(\phi) = \alpha$.

Lemma (2) If $E_\theta(\phi)$ is cont at $\theta \in \partial H_0$ $\forall$ test fn ϕ and ϕ_0 is UMP size α for ∂H_0 vs H_a and similar on ∂H_0 and of size α for H_0 vs H_a then ϕ_0 is UMPU size α for H_0 vs H_a.

Proof: Clearly ϕ_0 is unbiased for H_0 vs H_a since UMP size α for ∂H_0 vs H_a implies $E_\theta(\phi_0) \geq \alpha$ for $\theta \in H_a$ and of size α for H_0 vs H_a. If ϕ is unbiased size α for H_0 vs H_a then $E_\theta(\phi)$ is similar on ∂H_0 of size α and so $E_\theta(\phi) \leq E_\theta(\phi_0)$ $\forall \theta \in H_a$. which implies ϕ_0 is UMPU size α for H_0 vs H_a.

- so look for optimal similar size α test for ∂H_0 vs H_a and see if it has size α for H_0 vs H_a

Def A size α test for H_0 vs H_a has Neyman structure wrt ∂H_0 if $E_\theta(\phi \mid T) \equiv \alpha$ $\forall \theta \in \partial H_0$ when T is a mss for $\{f_\theta : \theta \in \partial H_0\}$

- if ϕ has Neyman structure then for $\theta \in \partial H_0$ $E_\theta(\phi) = E_\theta(E_\theta(\phi \mid T)) = \alpha$ and ϕ is similar on ∂H_0

- suppose every similar size α test has Neyman structure wrt T (later)

- let ϕ_0 be the MP size α test for $f_{\partial H_0}(\cdot|T)$ vs $f_{\theta_1}(\cdot|T)$ for some $\theta_1 \in H_a$ (apply FL)

- then for $\theta \in \partial H_0$, $E_\theta(\phi_0) = E_\theta(E_{\partial H_0}(\phi_0|T)) = \alpha$ and ϕ_0 is similar

- then if ϕ is a similar size α test, it has Neyman structure and so $E_{\theta_1}(\phi) = E_{\theta_1}(E_{\partial T}(\phi|T)) \leq E_{\theta_1}(\phi_0)$

- so ϕ_0 is the MP similar size α test for ∂H_0 vs $\{\theta_1\}$

- if ϕ_0 doesn't involve θ_1 then it is UMP similar size α for ∂H_0 vs H_a

- if $E_\theta(\phi)$ is cont at $\theta \in \partial H_0$ $\forall \phi$ and $E_\theta(\phi_0) \leq \alpha$ $\forall \theta \in H_0$ then ϕ_0 is UMPU size α for H_0 vs H_a, by Lemma 2

- when does every similar test have Neyman structure

<u>Lemma</u> 3 If T is a MSS for $\{f_\theta : \theta \in \partial H_0\}$ then all similar tests have Neyman structure iff T is boundedly complete for $\theta \in \partial H_0$.

Proof: ⇐] Suppose φ is similar of size α. Then for $\theta \in \partial H_0$

$$E_\theta(\varphi - \alpha) \overset{TTE}{=} E_\theta\left(E_{\partial H_0}(\varphi - \alpha)\mid T)(t)\right) \text{ and}$$

so $E_{\partial H_0}(\varphi - \alpha)\mid T)(t) = 0$ (a.e. $P_{\theta T}$).

⇒] Suppose T is not boundedly complete. Then $\exists\, h$ s.t. $|h(t)| \le M$ and $E_{\theta_T}(h(t)) = 0$ $\forall \theta \in \partial H_0$ and $P_{\theta_0 T}(\{t \mid h(t) \neq 0\}) > 0$ for some $\theta_0 \in \partial H_0$. Let $\varphi(t) = \frac{\min\{\alpha, 1-\alpha\}}{M} h(t) + \alpha$.

Then $0 \le h(t) \le 1$ and $E_\theta(\varphi) = \alpha$ $\forall \theta \in \partial H_0$. But

$E_{\theta_0}(\varphi \mid T)(t) = c\,h(t) + \alpha \neq \alpha$ in $\{t \mid h(t) \neq 0\}$

which contradicts φ having Neyman structure.

Eg - $x = (x_1, \dots, x_n) \sim N(\mu, \sigma^2)$

$\Theta = \mathbb{R}^1 \times (0, \infty)$, $H_0 : \sigma^2 \le \sigma_0^2$ vs $H_a : \sigma^2 > \sigma_0^2$

- no UMP size α test exists (Ex)

  - fix $\mu = \mu_0$

  - then UMP size α tests rejects for large values of $\sum_{i=1}^n (x_i - \mu_0)^2 / \sigma_0^2$ which depends on μ_0

- now $\sum_{i=1}^n (x_i - \bar{x})^2 / \sigma^2 \sim \chi^2_{n-1}$

- what about test

$$\varphi(x) = \begin{cases} 1 & \sum_{i=1}^n (x_i - \bar{x})^2 > \sigma_0^2 \chi^2_{n-1, 1-\alpha} \\ 0 & \text{otherwise} \end{cases}$$

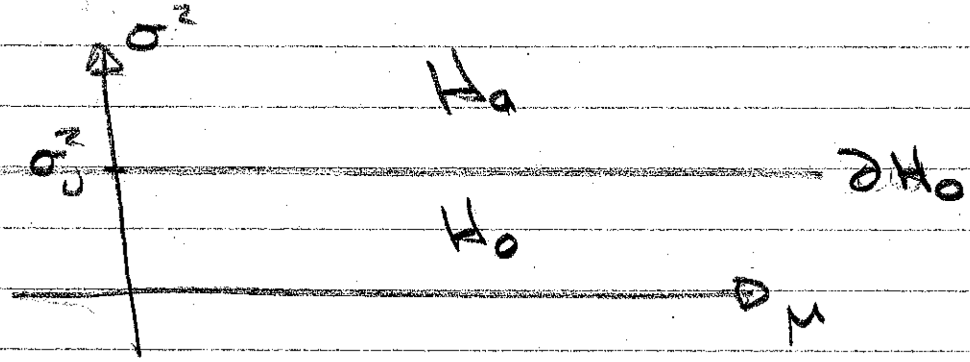
- $P_{(\mu,\sigma^2)}(\varphi(x) = 1)$

$$= P_{(\mu,\sigma^2)}\left(\sum_{i=1}^n (x_i - \bar{x})^2 / \sigma^2 > \left(\frac{\sigma_0^2}{\sigma^2}\right) \chi^2_{n-1,1-\alpha} \right)$$

$$= P\left(\chi^2_{n-1} > (\sigma_0^2/\sigma^2)\, \chi^2_{n-1,1-\alpha} \right)$$

$$\begin{cases} < \alpha & \text{when } \sigma^2 < \sigma_0^2 \quad \forall \mu \\ = \alpha & \sigma^2 = \sigma_0^2 \quad \forall \mu \\ > \alpha & \text{when } \sigma^2 > \sigma_0^2 \quad \forall \mu \end{cases}$$

so φ is unbiased, size α.



$s^2 \sim \text{Gamma}\left(\frac{n-1}{2}, \frac{(n-1)}{2\sigma^2}\right)$

- $(\bar{x}, s^2)$ is MSS (full problem)

  $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$ ind of $\frac{(n-1)s^2}{\sigma^2} \sim$ Chi-squared$(n-1)$ or

- restricted to ∂H_0 $\bar{x}$ is MSS and cond. dist. of $(\bar{x}, s^2)$ given $\bar{x}$ is that of s^2
- $\bar{x}$ is complete for ∂H_0 and thus boundedly complete
- pick $(\mu_1, \sigma_1) \in H_a$ we have that

  $\frac{(n-1)s^2}{\sigma_1^2} \sim$ Chi-squared$(n-1)$ (ind. of μ_1)

- therefore optimal test for ∂H_0 vs $\{(\mu_1,\sigma_1)\}$ rejects for large values of

$$\frac{\frac{1}{\Gamma(\frac{n-1}{2})}\left(\frac{n-1}{2\sigma_1^2}\right)^{\frac{n-1}{2}}(s^2)^{\frac{n-1}{2}-1}e^{-\left(\frac{n-1}{2\sigma_1^2}\right)s^2}}{\frac{1}{\Gamma(\frac{n-1}{2})}\left(\frac{n-1}{2\sigma_0^2}\right)^{\frac{n-1}{2}}(s^2)^{\frac{n-1}{2}-1}e^{-\left(\frac{n-1}{2\sigma_0^2}\right)s^2}}$$

or large values of (taking logs)

$$\left[-\left(\frac{n-1}{2\sigma_1^2}\right)+\left(\frac{n-1}{2\sigma_0^2}\right)\right]s^2 = \frac{1}{2}\left(1-\left(\frac{\sigma_0^2}{\sigma_1^2}\right)\right)\left(\frac{n-1}{\sigma_0^2}\right)s^2$$

equivalently large value of $\left(\frac{n-1}{\sigma_0^2}\right)s^2$

$\therefore \varphi(\bar{x}, s^2) = \begin{cases} 1 & \left(\frac{n-1}{\sigma_0^2}\right)s^2 \geq \chi^2_{1-\alpha}(n-1) \\ 0 & \text{otherwise} \end{cases}$

and note this does not involve (μ, σ_0^2)

- further when $\sigma^2 \leq \sigma_0^2$ we have that

  $P_{(\mu,\sigma^2)}\left(\left(\frac{n-1}{\sigma_0^2}\right)s^2 \geq \chi^2_{1-\alpha}(n-1)\right) \leq \alpha$ so it

  is of size α for H_0 vs H_a and is similar of size α on ∂H_0

- since $(\bar{x}, s^2)$ has a model of exponential form $E_{(\mu,\sigma^2)}(\varphi(\bar{x}, s^2))$ is continuous in $(\mu,\sigma^2) \in R \times [0,\infty)$ (natural parameter is μ/σ^2, $1/\sigma^2$) ($\frac{\partial}{\partial \mu} = \frac{\partial}{\partial m_1}\frac{\partial m_1}{\partial \mu} + \frac{\partial}{\partial m_2}\frac{\partial m_2}{\partial \mu}$, $\frac{\partial}{\partial \sigma^2} = \frac{\partial}{\partial m_1}\frac{\partial m_1}{\partial \sigma^2} + \frac{\partial}{\partial m_2}\frac{\partial m_2}{\partial \sigma^2}$)

- further $\bar{x}$ is complete for the model restricted to ∂H_0 and hence boundedly complete

- therefore φ is UMPU size α.

- UMPU tests for exponential models with one T function

- in natural parameterization.

$$f_\theta(x) = \exp\{\theta T(x) - A(\theta)\} h(x), \quad \theta \in \Theta$$

where Θ is a convex subset of $\mathbb{R}^1$

- has MLR property in $T(x)$ so when strictly MLR increasing

$$\varphi_1(x) = \begin{cases} 1 & T(x) > k_1 \\ \gamma_1 & = k_1 \\ 0 & < k_1 \end{cases}$$

is UMP size α for $H_0: \theta \le \theta_0$ vs $H_a: \theta > \theta_0$ (with appropriate choice of γ_1, k_1)

$$\varphi_2(x) = \begin{cases} 1 & T(x) < k_2 \\ \gamma_2 & = \\ 0 & > \end{cases}$$

is UMP size α for $H_0: \theta \ge \theta_0$ vs $H_a: \theta < \theta_0$ (with appropriate choice γ_2, k_2)

__Lemma__ (4) When $0 < \alpha < 1$ $\nexists$ a UMP size α test for $H_0 = \{\theta_0\}$ versus $H_a = \{\theta_0\}^c$.

Proof: Suppose such a φ exists. Then since φ_1 is MP size α for $\{\theta_0\}$ vs $\{\theta_1\}$ with $\theta_1 > \theta_0$ we must have $\varphi(x) = \varphi_1(x)$ provided $T(x) \ne k_1$ since φ

is also MP size α for $\{\theta_0\}$ vs $\{\theta_1\}$

and we apply FL(iii). By the Corollary

to the FL $\alpha = E_{\theta_0}(\varphi) = E_{\theta_0}(\varphi_1) < E_{\theta_1}(\varphi_1)$

$= E_{\theta_1}(\varphi)$ which means φ is not size α

for $\{\theta_0\}$ vs $\{\theta_0\}^c$.

- so we look for a UMPU size α test

Theorem (5) A UMPU size α test of $H_0 = \{\theta_0\}$ vs $H_a = \{\theta_0\}^c$ exists and it takes the form

$$\varphi(x) = \begin{cases} 1 & T(x) \notin (c_1, c_2) \\ \gamma_1 & T(x) = c_1 \\ \gamma_2 & T(x) = c_2 \\ 0 & \text{otherwise} \end{cases}$$

where the constants are chosen to satisfy
(i) $E_{\theta_0}(\varphi) = \alpha$ (ii) $E_\theta(\varphi T) = \alpha E_\theta(T)$

Proof: We can restrict attention to tests that are fns of the mss which has density $g_\theta(t) = c(\theta)\exp\{\theta t\}$ wrt support measure μ on $\mathcal{T}$. Suppose φ is an unbiased test. Then $\partial H_0 = \theta_0$ and since $E_\theta(\varphi)$ is continuous at θ_0 we have $E_{\theta_0}(\varphi) = \alpha$.

and so θ_0 is a minimum of $E_\theta(\varphi)$. Since $E_\theta(\varphi)$ is differentiable at θ_0 then

$$0 = \frac{\partial E_\theta(\varphi)}{\partial\theta}\bigg|_{\theta=\theta_0} = \frac{\gamma'(\theta)}{\gamma(\theta)} E_{\theta_0}(\varphi) + E_{\theta_0}(T\varphi).$$

Now $\gamma'(\theta) = \frac{\partial}{\partial\theta}\left(\int_{\mathcal{X}} \exp\{\theta t\}\mu(dt)\right)^{-1}$

$= -(\ \ )^{-2}\int_{\mathcal{X}} t\exp\{\theta t\}\mu(dt) = -\gamma(\theta)E_\theta(T).$

and so $0 = -E_{\theta_0}(T)E_{\theta_0}(\varphi) + E_{\theta_0}(T\varphi)$

which implies $E_{\theta_0}(T\varphi) = \alpha E_{\theta_0}(T)$.

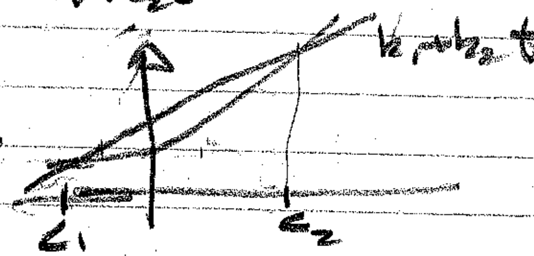
Set $m=2$, $f_0(t) = g_{\theta_1}(t)$, $f_1(t) = g_{\theta_0}(t)$, $f_2(t) = t g_{\theta_0}(t)$, note that $f_0(t) \gtreqless k_1 f_1(t) + k_2 f_2(t)$ iff $g_{\theta_1}(t) \gtreqless (k_1 + k_2 t) g_{\theta_0}(t)$ and by the GFL the corresponding test φ_0 maximizes $E_{\theta_1}(\varphi)$ among all tests φ satisfying $E_{\theta_0}(\varphi) \le E_{\theta_0}(\varphi_0)$, $E_{\theta_0}(\varphi) \le E_{\theta_0}(T\varphi)$.

Assume hereafter (Lemma 6) $\exists k_1, k_2$ & s.t. φ_0 satisfies (i) and (ii) and so this test is MP size α for $\{\theta_0\}$ vs $\{\theta_1\}$ among all tests satisfying (i) and (ii) (restrict inequalities to equalities).

Now $g_{\theta_1}(t)/g_{\theta_0}(t) \gtreqless k_1 + k_2 t$ (*) iff

$$\frac{\gamma(\theta_1)}{\gamma(\theta_0)} \exp\{t(\theta_1 - \theta_0)\} \gtreqless k_1 + k_2 t$$

and choosing $\theta_1 > \theta_0$ implies



there are two solutions c_1, c_2 s.t. $t \notin (c_1, c_2)$, $t \notin \{c_1, c_2\}$, $t \in (c_1, c_2)$ sats (*) (otherwise φ_0 would be to reject $\{\theta_0\}$ for large values of t which is the UMP size α test for $H_0: \theta \leq \theta_0$ vs $H_a: \theta > \theta_0$ and this cannot satisfy (ii) as $E_\theta(\varphi_0)$ is increasing in θ).

Similarly if $\theta_1 < \theta_0$ and so the test must be of the form stated in the theorem. Since conditions (i) and (ii) depend only on θ_0

the test does not involve θ_1 and so φ_0 is UMP size α for $\{\theta_0\}$ vs $\{\theta_0\}^c$ among all tests satisfying (i) and (ii). Since this includes all unbiased tests this proves φ_0 is UMPU size α for $\{\theta_0\}$ vs $\{\theta_0\}^c$.

- <u>note</u> γ_1, γ_2 must satisfy

$$\gamma_1 P_{\theta_0}(T(x) = c_1) + \gamma_2 P_{\theta_0}(T(x) = c_2) = \gamma P_{\theta_0}(T(x) \in \{c_1, c_2\})$$

and are otherwise arbitrary.

<u>Lemma 6</u> A test exists of the form specified in Thm 6

Proof: Let ϕ_u be the UMP size u test for $\theta \le \theta_0$ vs $\theta > \theta_0$ so it is of the form

$\phi_u(t) = \begin{cases} 1 & t > c_u \\ \gamma_u & = \\ 0 & < \end{cases}$. Now for $0 \le u \le \alpha$ put

$\phi^*_u(t) = \phi_u(t) + 1 - \phi_{1-\alpha+u}(t)$ then since

$u \le 1-\alpha+u$ we have $c_{1-\alpha+u} \le c_u$,

$$\phi^*_u(t) = \begin{cases} 1 & t \notin (c_{1-\alpha+u}, c_u) \\ \gamma_{1-\alpha+u} & t = c_{1-\alpha+u} \\ \gamma_u & t = c_u \\ 0 & \text{otherwise} \end{cases}$$

and $E_{\theta_0}(\phi^*_u(t)) = u + 1 - (1-\alpha+u) = \alpha$

so ϕ^*_u is size α for $\{\theta_0\}$ vs $\{\theta_0\}^c$.

Note $\phi^*_0 = \phi_0 + 1 - \phi_{1-\alpha} = 0 + 1 - \phi_{1-\alpha} = 1 - \phi_{1-\alpha}$

$\phi^*_\alpha = \phi_\alpha + 1 - \phi_1 = \phi_\alpha + 1 - 1 = \phi_\alpha$.

The size α test that maximizes

$\dfrac{\partial E_\theta(\phi)}{\partial \theta}\Big|_{\theta=\theta_0}$ is the LMP size α test

of $\{\theta_0\}$ vs $\{\theta_1\}$ with $\theta_1 > \theta_0$ is of the form

$S'(\theta_0|x) = \frac{\gamma'(\theta_0)}{\gamma(\theta_0)} + t \gtrless \hat{n}$, which is to reject for large values of t and so is ϕ_α. Similarly the test which minimizes

$$\frac{\partial E_\theta(\phi)}{\partial\theta}\Big|_{\theta=\theta_0}$$ is the LMP size α

test of $\{\theta_0\}$ vs $\{\theta_1\}$ for $\theta_1 < \theta_0$ and this rejects for $t < c$ and so is $1-\phi_{1-\alpha}$.

By comparison with $\phi(t) \equiv \alpha$ we have that

$$\frac{\partial E_\theta(1-\phi_{1-\alpha})}{\partial\theta}\Big|_{\theta=\theta_0} \le 0 \le \frac{\partial E_\theta(\phi_\alpha)}{\partial\theta}\Big|_{\theta=\theta_0}$$

If we prove that $\frac{\partial E_{\theta_0}(\phi_u^*)}{\partial u}$ is continuous in u then the Intermediate Value Thm gives the result.

So put $h(u) = \frac{\partial}{\partial\theta} E_\theta(\phi_u)\Big|_{\theta=\theta_0}$

and note $\frac{\partial}{\partial\theta} E_{\theta_0}(\phi^*)\Big|_{\theta=\theta_0} = h(u) - h(1-\alpha+u)$ and so it is enough to prove h is cont.

Put $F_{\theta_0}(t,v) = P_{\theta_0,T}((t,\infty)) + v P_{\theta_0,T}(\{t\})$

Then by Lemma ② (below) $W = F_{\theta_0}(t,v)$

$\sim U(0,1)$ when $t \sim P_{\theta_0,T}$ stat. ind. of $v \sim U(0,1)$.

and recall that c_u is s.t. $u = P_{\theta_0}(T > c_u) + \gamma_u P_{\theta_0}(T = c_u)$

$= F_{\theta_0}(c_u, \gamma_u)$ so $P_{\theta_0}(F_{\theta_0}(t,v) < u \mid T)(t)$

$$= \begin{cases} 0 & t < c_u \\ \gamma & t = c_u \\ 1 & t > c_u . \end{cases}$$

Now $h(u) = \left.\dfrac{\partial E_\theta(\alpha)}{\partial \theta}\right|_{\theta=\theta_0} \overset{\text{previously shown}}{=} E_{\theta_0}(T\alpha_u) - u E_{\theta_0}(T)$

$= E_{\theta_0}(T(\alpha_u - u)) \overset{TTE}{=} E(T(I_{[0,u)}(W) - u))$

Therefore $\lim_{\varepsilon \to 0} g(u+\varepsilon) - g(u) = E(T(I_{[0,u+\varepsilon)}(W) - I_{[0,u)}(W)))$

$\overset{DCT}{=} E(I_{\{u\}}(W)T) = 0$ since $P(W=u) = 0$.

and h is continuous.

- we now need to prove $W \sim U(0,1)$

- let $X \sim P$ and put $F(x,v) = P(X > x) + v\,P(\{x\})$

- say $(x,v) < (x',v')$ iff $x > x'$ or $x = x'$, $v < v'$
  so F is nondecreasing in (x,v)

- $\lim_{x \to \infty} F(x,v) = 0$, $\lim_{x \to -\infty} F(x,v) = 1$ and F
  assumes every value in $(0,1)$

<u>**Lemma ⑦**</u> If $X \sim P$ stat. ind of $V \sim U(0,1)$, then
$W = F(X,V) \sim U(0,1)$.

Proof: Let $w_0 \in (0,1)$ and let (x_0, v_0) be st.

$w_0 = F(x_0, v_0)$. Then $P(W \le w_0)$

$= P(F(X,V) \le F(x_0,v_0))$

$= P(F(X,V) < F(x_0,v_0)) + P(F(X,V) = F(x_0,v_0))$

$= P((X,V) < (x_0,v_0)) + 0$ (Lemma ⑧)

$= P(X > x_0) + P(V < v_0)P(X = x_0) = F(x_0, v_0) = w_0$

and the result is proved.

<u>**Lemma ⑧**</u> $P(F(X,V) = F(x_0,v_0)) = 0$

Proof: Suppose $F(x,v) = F(x_0,v_0)$. If $x < x_0$

then $P(X > x) \ge P(X \ge x_0) \ge F(x_0,v_0)$

and so $P(X=x) = P(X=x_0) = 0$. Therefore

$G(x) = P(X \le x) \ge P(X \le x_0) = G(x_0)$ and let $x_{00} = \inf\{x : G(x) = G(x_0)\}$.

Then $P(X < x_0, F(X,V) = F(x_0, v_0))$

$= P(X \in [x_{00}, x_0]) = G(x_0) - G(x_{00}) = 0$

since $G(x_{00}) = G(x_0)$ and $P(X = x_{00}) = 0$.

The argument when $x > x_0$ is the same and when $x = x_0$ $P(V = v_0) = 0$.

Eg $x = (x_1, \dots, x_n) \sim$ iid $N(\mu, \sigma_0^2)$ $\mu \in R^1$

- $\bar{x}$ a MSS, $H_0 = \{\mu_0\}$ vs $H_a = \{\mu_0\}^c$

- $$\phi(\bar{x}) = \begin{cases} 1 & x \notin (c_1, c_2) \\ 0 & \bar{x} \in (c_1, c_2) \end{cases}$$

- determine c_1, c_2 via

(i) $\alpha = \int_{-\infty}^{\infty} \phi(\bar{x}) f_{\mu_0, \frac{\sigma_0^2}{n}}(\bar{x})\, d\bar{x} = 1 - \left(\Phi\left(\sqrt{n}\left(\frac{c_2 - \mu_0}{\sigma_0}\right)\right) - \Phi\left(\sqrt{n}\frac{(c_1 - \mu_0)}{\sigma_0}\right)\right)$

(ii) $\alpha \mu_0 = \int_{-\infty}^{\infty} \bar{x}\, \phi(\bar{x}) f_{\mu_0, \frac{\sigma_0^2}{n}}(\bar{x})\, d\bar{x} = \mu_0 - \int_{c_1}^{c_2} \bar{x}\, f_{\mu_0, \frac{\sigma_0^2}{n}}(\bar{x})\, d\bar{x}$

put $u = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma_0}$ $du = \frac{\sqrt{n}}{\sigma_0} d\bar{x}$

$$\bar{x} = \mu_0 + \frac{\sigma_0}{\sqrt{n}} u$$

$$= \mu_0 - \int_{\frac{\sqrt{n}(c_1 - \mu_0)}{\sigma_0}}^{\frac{\sqrt{n}(c_2 - \mu_0)}{\sigma_0}} \left(\mu_0 + \frac{\sigma_0}{\sqrt{n}} u\right) f_{0,1}(u)\, du$$

$$= \mu_0 - \mu_0(1-\alpha) + \frac{\sigma_0}{\sqrt{n}} \int_{\frac{\sqrt{n}(c_1 - \mu_0)}{\sigma_0}}^{\sqrt{n}\left(\frac{c_2 - \mu_0}{\sigma_0}\right)} u\, f_{(0,1)}(u)\, du$$

$$\therefore 0 = \int_{\sqrt{n}\left(\frac{c_1 - \mu_0}{\sigma_0}\right)}^{\sqrt{n}\left(\frac{c_2 - \mu_0}{\sqrt{n}}\right)} u\, f_{(0,1)}(u)\, du = -\int_{\sqrt{n}\left(\frac{c_1 - \mu_0}{\sigma_0}\right)}^{\sqrt{n}\left(\frac{c_2 - \mu_0}{\sigma_0}\right)} \frac{\partial f_{0,1}(u)}{\partial u}\, du$$

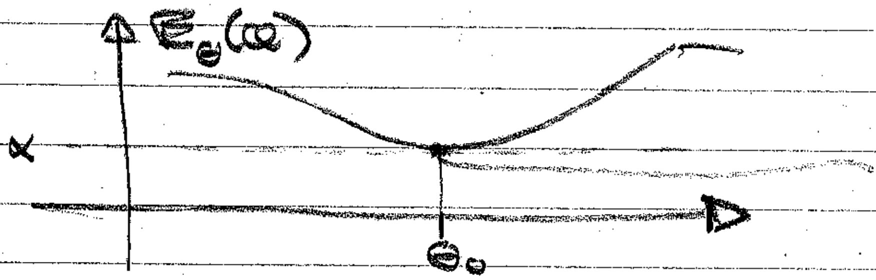
$$= f_{0,1}\left(\sqrt{n}\left(\frac{c_2 - \mu_0}{\sigma_0}\right)\right) - f_{0,1}\left(\sqrt{n}\left(\frac{c_1 - \mu_0}{\sigma_0}\right)\right)$$

$$\therefore \sqrt{n}\left(\frac{c_2 - \mu_0}{\sigma_0}\right) = -\sqrt{n}\left(\frac{c_1 - \mu_0}{\sigma_0}\right) \text{ and } c_1 = -c_2 = z_{1-\alpha/2}$$

- locally MP unbiased tests

- suppose a UMPU size α test for $H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$ doesn't exist

- looking among unbiased tests for something with an optimality property



want this to be a local minimum

- relax to locally unbiased (θ_0) size α tests

$$E_{\theta_0}(\phi) = \alpha$$

$$0 = \left.\frac{\partial E_\theta(\phi)}{\partial \theta}\right|_{\theta_0} \overset{\text{under conditions}}{=} E_{\theta_0}\left(\phi(x) S(x|\theta_0)\right)$$

- I want max

$$\left.\frac{\partial^2 E_\theta(\phi)}{\partial \theta^2}\right|_{\theta=\theta_0}$$

$$\overset{\text{under condition}}{=} E_{\theta_0}\left(\phi(x)\left[\left.\frac{\partial S(x|\theta)}{\partial \theta}\right|_{\theta_0} + S^2(x|\theta_0)\right]\right)$$

- using generalized FL

put $f_0(x) = \left(\frac{\partial S(x|\theta)}{\partial\theta}\Big|_{\theta=\theta_0} + S^2(x|\theta_0) \right) f_{\theta_0}(x)$

$$f_1(x) = S(x|\theta_0)\, f_{\theta_0}(x)$$

$$f_2(x) = f_{\theta_0}(x)$$

- so test is of the form

$$\begin{array}{ll} 1 & \left(\frac{\partial S(x|\theta)}{\partial\theta}\right)\Big|_{\theta=\theta_0} + S^2(x|\theta_0) > k_1 S(x|\theta_0) + k_2 \\ \gamma & \\ 0 & \qquad\qquad\qquad\qquad < \end{array}$$

for some k_1, k_2, γ determined (if possible) so that size is α and 1st derivative of power at θ_0 is 0.

supplemental

(a) UMPU tests for multiparameter exponential families

— $f_\theta(s) = \gamma(\theta) \exp\{\sum_{i=1}^{r} a_i(s)\psi_i(\theta)\} h(s)$

— natural parameterization

$$f_{\underset{\sim}{\theta}}(s) = \gamma(\underset{\sim}{\theta}) \exp\{\sum_{i=1}^{r} a_i(s)\theta_i\} h(s)$$

— consider the testing problems

$H_0^1 : \theta_1 \leq \theta_0$ vs $A^1 : \theta_1 > \theta_0$
$H_0^2 : \theta_1 \geq \theta_0$ vs $A^2 : \theta_1 < \theta_0$
$H_0^3 : \theta_1 = \theta_0$ vs $A^3 : \theta_1 \neq \theta_0$

— for these problems we can restrict to tests which are fns of the MSS $(a_1, \ldots, a_r)$

— put $t = (a_2, \ldots, a_r)$, then the cond'l dist of a_1 given t has density

$$g_{\theta_1}(a_1 | t) = \gamma_t(\theta_1) \exp\{a_1\theta_1\} h_t(a)$$

with respect to the relevant support measure ν.

— then intuitively we would expect optimal tests to be given by

$$\phi_1(a_1, t) = \begin{cases} 1 & a_1 > c_1(t) \\ \gamma_1(t) & = \\ 0 & < \end{cases}$$

$$\phi_2(a_1,t) = \begin{cases} 1 & a_1 < c_2(t) \\ \gamma_2(t) & \quad = \\ 0 & \quad > \end{cases}$$

$$\phi_3(a_1,t) = \begin{cases} 1 & a_1 \notin (c_{31}(t),\ c_{32}(t)) \\ \gamma_{31}(t) & a_1 = c_{31}(t) \\ \gamma_{32}(t) & a_1 = c_{32}(t) \\ 0 & \text{otherwise} \end{cases}$$

Theorem (8) The tests ϕ_1, ϕ_2, ϕ_3 are UMPU at their respective sizes and the constants can be chosen to give size α.

Proof: Given t the model for a_1 is of exp. form with 1 parameter and thus by the results of earlier sections the conditional tests ϕ_1, ϕ_2 are UMP at their respective sizes and the constants can be chosen so as to give size α. Further the conditional test ϕ_3 is UMPU at its size and the constants can be chosen to give size α and $E_{\theta_0}[\phi_3(a_1,t)a_1|t] = \alpha E_{\theta_0}[a_1|t]$

We must now show that these tests are UMPU of size α unconditionally.

(1) Consider first ϕ_1. Since $P_{\phi_1}(\theta_1|t) \begin{matrix} \geq \alpha & \theta_1 \geq \theta_0 \\ \leq \alpha & \theta_1 \leq \theta_0 \end{matrix}$

Then $P_{\phi_1}(\underset{\sim}{\theta}) = \underset{\underset{\sim}{\theta}}{\overset{t}{E}}[P_{\phi_1}(\theta_1|t)] \begin{matrix} \geq & \underset{\sim}{\theta} \in H' \\ \leq & \underset{\sim}{\theta} \in H_0' \end{matrix}$

and ϕ_1 is unconditionally unbiased of size α.

The continuity of the power fn implies that $\alpha = P_{\phi_1}(\underset{\sim}{\theta})$ $\forall \underset{\sim}{\theta} \in \omega = \{\theta \mid \pi_1(\underset{\sim}{\theta}) = \theta_0\}$, i.e. ϕ_1 is similar on ω. The restricted model $\{P_{\underset{\sim}{\theta}} \mid \underset{\sim}{\theta} \in \omega\}$ is of exponential form with $(r-1)$ π fns and thus $t = (a_2, \ldots, a_r)$ is a MSS and it is complete.

Then all similar tests have Neyman structure; i.e. conditionally given t, any test similar on ω has conditional size α. Now conditionally ϕ_1 is UMP of size α for $\theta_1 = \theta_0$ vs $\theta_1 > \theta_0$; i.e it maximizes $P_\phi(\theta_1|t)$ for each $\theta_1 > \theta_0$

Then by TTE we must have that a_1 maximizes $P_a(\underset{\sim}{\theta})$ for every $\underset{\sim}{\theta} \in A$. Thus a_1 is unconditionally uniformly most powerful amongst all tests similar on ω of size α for ω in A. By the cty of the power fn any unbiased test of size α is similar on ω of size α and since a_1 is unconditionally unbiased a_1 is unconditionally UMPU of size α.

(2) The proof that a_2 is unconditionally UMPU of size α follows as in (1).

(3) Consider now a_3. Now suppose that a is unconditionally unbiased of size α for H_0^3 in A^3. Then for fixed $\theta_2, \ldots, \theta_r$, $\underset{\sim}{\theta}_0 = (\theta_0, \ldots, \theta_r)'$ is a min pt which implies that $\frac{\partial}{\partial \theta_1} P_a(\underset{\sim}{\theta})\Big|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} = 0$ which implies $E_{\underset{\sim}{\theta}_0}[a \cdot a_1] = \alpha E_{\underset{\sim}{\theta}_0}[a]$

Problem. Since t is complete this implies $E_{\theta_0}[\phi \cdot a_1 | t] = \alpha E_{\theta_0}[a_1 | t]$ (2) except perhaps on a set of t values having $P_{\theta_0}^{t}$ measure 0 ⊕ $\theta_0 \in \omega$. Further as in (1) ϕ has Neyman structure and thus $E_{\theta_0}[\phi | t] = \alpha$ (1). Now ϕ_3 is conditionally UMP amongst all tests satisfying (1) and (2). Then as in (1) ϕ_3 is UMP unconditionally amongst all tests satisfying the unconditional versions of (1) and (2) (which ϕ_3 does). Further ϕ_3 is unbiased since conditionally it has power greater than the test $\phi(a) \equiv \alpha$. Therefore ϕ_3 is UMPU of size α

Note ① — there is trouble in the above proof on assumption that the fns ϕ, ϕ_2 & ϕ_3 are measurable — see Lehmann for proof

② – we also obtain UMPU tests for $\theta^* = \underset{\sim}{l}'\underset{\sim}{\theta}$ where $\underset{\sim}{l} \in R^r - \{\underset{\sim}{0}\}$ is known

Proof: $f_\theta(s) = \gamma(\underset{\sim}{\theta}) \exp\{\sum_{i=1}^r a_i(s)\theta_i\} h(s)$

wlog $l_1 \neq 0$

$= \gamma(\underset{\sim}{\theta}^*) \exp\{\sum_{i=1}^r a_i^*(s)\theta_i^*\} h(s)$ where

$\theta_i^* = \theta_i \quad i=2,\dots,r \quad ; \quad a_1^*(s) = a_1(s)/l_1,$

$a_i^*(s) = a_i(s) - l_i a_1(s)/l_1 \quad i=2,\dots,r$

③ – the cond'l power $P_\alpha(\theta_1|t)$ does not depend upon $(\theta_2,\dots,\theta_r)$ since the cond'l dist. of $a_1|t$ is ind. of $(\theta_2,\dots,\theta_r)$

– hence the unconditional power fn does

– note that $P_\alpha(\theta_1|t)$ is an unbiased est'r of $P_\alpha(\underset{\sim}{\theta})$ and since t is complete it is the only one.

Eg – $s = (x_1,\dots,x_n)'$ is a sample from $N(\mu,\sigma^2)$ where (μ,σ^2) is unknown

– suppose we want UMPU size α test for $H_0: \mu = \mu_0$ vs $A: \mu \neq \mu_0$

– $f_\theta(s) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\}$

$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0+\mu_0-\mu)^2\}$

$$= \left[(2\pi\sigma^2)^{-n/2} \exp\left\{ \frac{-n}{2\sigma^2}(\mu_0-\mu)^2 \right\} \right] \cdot \exp\left\{ n\bar{y}\left(\frac{\mu_0-\mu}{\sigma^2}\right) - \left(\sum_{i=1}^{n} y_i^2\right)\frac{1}{2\sigma^2} \right\}$$

where $y_i = x_i - \mu_0 \sim N(\mu-\mu_0, \sigma^2)$

— clearly testing $H_0: \mu = \mu_0$ vs $A: \mu \neq \mu_0$ is equivalent to testing $H_0: \frac{n(\mu-\mu_0)}{\sigma^2} = 0$ vs $A: \frac{n(\mu-\mu_0)}{\sigma^2} \neq 0$

— by the Thm this is given by a test of the form $\varphi(s) = 1$ whenever $\bar{y} \notin (c_1(\sum y_i^2), c_2(\sum y_i^2))$ given $\sum y_i^2$ and the constants are chosen s.t.

$$E_{(\mu_0,\sigma^2)}[\varphi \mid \sum y_i^2] = \alpha \quad (1), \quad E_{(\mu_0,\sigma^2)}[\varphi\,\bar{y} \mid \sum y_i^2] = \alpha E_{(\mu_0,\sigma^2)}[\bar{y} \mid \sum y_i^2] \quad (2)$$

— now given $\sum y_i^2$, $t = \sqrt{n}\,\bar{y}/s_y^2$ where $s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2 = \frac{1}{n-1}\left(\sum y_i^2 - n\bar{y}^2\right) = s_x$ is a 1-1 increasing fn of $\bar{y}$ Problem so reject whenever t is outside some interval

— the cond'l dist'n of s given $\sum y_i^2$ under H_0 is uniform on the sphere of radius $\sqrt{\sum y_i^2}$ o Problem

— the cond'l dist'n of t given $\sum y_i^2$ under H_0 is Student $(n-1)$ Problem and this is also the unconditional dist of t and thus t is independent of $\sum y_i^2$

— under H_0, $\bar{y} \sim N(0, \sigma^2/n)$ and thus $E_{(\mu_0,\sigma^2)}[\bar{y}] = 0$ which implies $E_{(\mu_0,\sigma^2)}[\bar{y} \mid \sum y_i^2] = 0$ since $\sum y_i^2$ is complete and thus (2) takes the form $E_{(\mu_0,\sigma^2)}[\varphi \cdot \bar{y} \mid \sum y_i^2] = 0$