STA 3000 : 2020 Exercises 1 - Solutions

1) (a) $R(a,d) = \frac{1}{4}L(a,a) + \frac{1}{4}L(a,a) + 0 \cdot L(a,a) + \frac{1}{2}L(a,b)$

$= \frac{1}{2}$

$R(b,d) = \frac{1}{2}L(b,a) + 0L(b,a) + \frac{1}{4}L(b,a) + \frac{1}{4}L(b,b)$

$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

b) Consider the dec. fn given by $d_0(1) = b$, $d_0(2) = a$, $d_0(3) = b$, $d_0(4) = a$. Then

$R(a,d_0) = \frac{1}{4}L(a,b) + \frac{1}{4}L(a,a) + 0L(a,b) + \frac{1}{2}L(a,a)$

$= \frac{1}{4}$

$R(b,d_0) = \frac{1}{2}L(b,b) + 0L(b,a) + \frac{1}{4}L(a,b) + \frac{1}{4}L(a,a)$

$= \frac{1}{4}$

Therefore $R(\theta,d_0) < R(\theta,d)$ $\forall\theta$ and d is not admissible.

(c) $r(d) = \frac{1}{4}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{3}{4} = \frac{1}{8} + \frac{9}{16} = \frac{11}{16}$

(d) We first need to determine the posterior distribution of θ for each data value.

Posterior distribution

$x=1$ $\quad \theta=a \quad \frac{1}{4}\cdot\frac{1}{4} \Big/ \left(\frac{1}{4}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{2}\right) = 1/7$

$\qquad\;\; \theta=b \quad \frac{3}{4}\cdot\frac{1}{2} \Big/ \left(\frac{1}{4}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{2}\right) = 6/7$

$x=2$ $\quad \theta=a \quad \frac{1}{4}\cdot\frac{1}{4} \Big/ \frac{1}{4}\cdot\frac{1}{4} = 1$

$\qquad\;\; \theta=b \quad \frac{3}{4}\cdot 0 \Big/ \frac{1}{4}\cdot\frac{1}{4} = 0$

$x=3$ $\quad \theta=a \quad \frac{1}{4}\cdot 0 \Big/ \frac{3}{4}\cdot\frac{1}{4} = 0$

$\qquad\;\; \theta=b \quad \frac{3}{4}\cdot\frac{1}{4} \Big/ \frac{3}{4}\cdot\frac{1}{4} = 1$

$x=4$ $\quad \theta=a \quad \frac{1}{4}\cdot\frac{1}{2} \Big/ \left(\frac{1}{4}\cdot\frac{1}{2}+\frac{3}{4}\cdot\frac{1}{4}\right) = 2/5$

$\qquad\;\; \theta=b \quad \frac{3}{4}\cdot\frac{1}{4} \Big/ \left(\frac{1}{4}\cdot\frac{1}{2}+\frac{3}{4}\cdot\frac{1}{4}\right) = 3/5$

Now $\sum_a L(\theta,a)\delta(x,da) = \delta(x,\{a\})L(\theta,a)+\delta(x,\{b\})L(\theta,b)$

$= \delta(x,\{a\})L(\theta,a) + (1-\delta(x,\{a\}))L(\theta,b)$

When $x=1$ posterior risk is

$(1-\delta(1,\{a\}))\frac{1}{7} + \delta(1,\{a\})\frac{6}{7} = \frac{1}{7}+\frac{5}{7}\delta(1,\{a\})$

& this is minimized by taking $\delta(1,\{a\})=0$

When $x=2$ posterior risk is

$(1-\delta(2,\{a\}))\cdot 1 + \delta(2,\{a\})\, 0 = 1-\delta(2,\{a\})$

& this is minimized by taking $\delta(2,\{a\})=1$

When $x=3$ posterior risk is

$$(1-\delta(3,\{a\}))0 + \delta(3,\{a\})1 = \delta(3,\{a\})$$

and this is minimized by taking $\delta(3,\{a\}) = 0$.

When $x=4$ posterior risk is

$$(1-\delta(4,\{a\}))\frac{2}{5} + \delta(4,\{a\})\frac{3}{5} = \frac{2}{5} + \delta(4,\{a\})\frac{1}{5}$$

and this is minimized by taking $\delta(4,\{a\}) = 0$.

Therefore the Bayes rule is nonrandomized and is given by

$d(1) = b$ $\quad$ $d(3) = b$

$d(2) = a$ $\quad$ $d(4) = b$

(e) $\int_{\{a,b\}} L(\theta, d) \, \delta(x, d)$

$\overset{x=1}{=} \frac{1}{2} L(\theta, a) + \frac{1}{2} L(\theta, b) = \frac{1}{2}$

$\overset{x=2}{=} \frac{1}{2} L(\theta, a) = \begin{cases} 0 & \theta = a \\ \frac{1}{2} & \theta = b \end{cases}$

$\overset{x=3}{=} \frac{1}{3} L(\theta, a) + \frac{2}{3} L(\theta, b) = \begin{cases} \frac{2}{3} & \theta = a \\ \frac{1}{3} & \theta = b \end{cases}$

$\overset{x=4}{=} \frac{2}{3} L(\theta, a) + \frac{1}{3} L(\theta, b) = \begin{cases} \frac{1}{3} & \theta = a \\ \frac{2}{3} & \theta = b \end{cases}$

$R(a, \delta) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot 0 + 0 \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{7}{24}$

$R(b, \delta) = \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{2}$

$r(\delta) = \frac{1}{4} \cdot \frac{7}{24} + \frac{3}{4} \cdot \frac{1}{2} = \frac{43}{96}$

(f) $d_0 < \delta < d$

2) (b) $\int_a L(\theta,a)\, \delta_p(x,da) = p L(\theta, x-1) + (1-p) L(\theta, x+1)$

and so $R(\theta, \delta_p) = p \int_x L(\theta, x-1) P_\theta(dx) + (1-p) \int_x L(\theta, x+1) P_\theta(dx)$

$= p\left(L(\theta, \theta-2)\frac{1}{2} + L(\theta,\theta)\frac{1}{2}\right) + (1-p)\left(\frac{1}{2}L(\theta,\theta) + \frac{1}{2}L(\theta,\theta+2)\right)$

$= p\frac{1}{2} + (1-p)\frac{1}{2} = \frac{1}{2}$

(a) $\int_a L(\theta,a)\,\delta(x,da) = L(\theta, d(x))$

and so $R(\theta, d) = \int_x L(\theta, d(x))\, P_\theta(dx)$

$= \sum_{x<0} L(\theta, x+1) P_\theta(\{x\}) + \sum_{x\geq 0} L(\theta, x-1) P_\theta(\{x\})$

$$= \begin{cases} \frac{1}{2} & \theta < -1 \\ \frac{1}{2} & \theta \geq 1 \\ 0 & \theta \in \{-1, 0\} \end{cases}$$

(c) So we prefer d to δ_p since

$R(\theta,d) \leq R(\theta,\delta_p) \quad \forall \theta$ and

$R(0,d) < R(0,\delta_p)$.

The posterior risk is proportional to

$$\int_{-\infty}^{\infty} (\theta - d(x))^2 \exp\{-\tfrac{1}{2}[|x-\theta| + |\theta - m|\} \, d\theta$$

Case 1 $m \le x$

$$m \qquad\qquad x$$

$$= \int_{-\infty}^{m} (\theta - d(x))^2 \exp\{-\tfrac{1}{2}[(x - 2\theta + m)]\} \, d\theta$$

$$+ \int_{m}^{x} (\theta - d(x))^2 \exp\{-\tfrac{1}{2}(x-m)\} \, d\theta$$

$$+ \int_{x}^{\infty} (\theta - d(x))^2 \exp\{-\tfrac{1}{2}(2\theta - x - m)\} \, d\theta$$

$$= \exp\{-\tfrac{1}{2}(x+m)\} \int_{-\infty}^{m} (\theta - d(x))^2 e^{\theta} \, d\theta \; +$$

$$\exp\{-\tfrac{1}{2}(x-m)\} \int_{m}^{x} (\theta - d(x))^2 \, d\theta \; +$$

$$\exp\{-\tfrac{1}{2}(-x-m)\} \int_{x}^{\infty} (\theta - d(x))^2 e^{-\theta} \, d\theta$$

$$= \exp\{-\tfrac{1}{2}(x+m)\} e^{m} [m^2 - 2m + 2 - 2md + 2d + d^2]$$

$$+ \exp\{-\tfrac{1}{2}(x-m)\} [\tfrac{1}{3}x^3 - x^2 d + x d^2 - \tfrac{1}{3}m^3 + m^2 d - m d^2]$$

$$+ \exp\{-\tfrac{1}{2}(-x-m)\} e^{-x} [x^2 + 2x + 2 - 2xd + 2d + d^2]$$

$$= \exp\{-\tfrac{1}{2}x + \tfrac{1}{2}m\} [(1 + x - m + 1)d^2 + (-2m + \cancel{2} - x^2 + m^2 - 2x + \cancel{2})d + C]$$

$$= \exp\{-\tfrac{1}{2}x + \tfrac{1}{2}m\} [A d^2 + B d + C] \quad \text{min'd for}$$

$$d = -\frac{B}{2A} = \frac{x^2 + 2x - m^2 + 2m}{2(x - m + 2)} = \frac{1}{2} \frac{(x - m + 2)(x + m)}{(x - m + 2)} = \frac{x + m}{2}$$

$$= \exp\{-\tfrac{1}{2}x + \tfrac{1}{2}n\}\Big[(1+x-n+1)d^2 + (-2n+2-x^2+n^2-2x-2)d + (\;)\Big]$$

$$= \exp\{-\tfrac{1}{2}x + \tfrac{1}{2}n\}\left[Ad^2 + Bd + c\right]$$

min'd by

$$d = -\frac{B}{2A} = \frac{x^2+2x-n^2+2n}{2(x-n+2)}$$

$$= \frac{1}{2}\,\frac{(x-n+2)(x+n)}{(x-n+2)} = \frac{x+n}{2}$$

The same reasoning produces this answer for Case 2: $n \geq x$.