STA 3000 Exercise 3 – Solutions

1. (a) $LR(\bar{x}) = \exp\{-\frac{n}{2}[(\bar{x}-\mu_1)^2 - (\bar{x}-\mu_0)^2]\}$

$= \exp\{-\frac{n}{2}[-2(\mu_1-\mu_0)\bar{x} + (\mu_1^2-\mu_0^2)]\}$

$P_{\mu_i}(LR(\bar{x}) > 1) = P_{\mu_i}\left(\bar{x} > \frac{(\mu_1^2-\mu_0^2)}{2(\mu_1-\mu_0)}\right)$ (since $\mu_1 > \mu_0$)

$= P_{\mu_i}\left(\bar{x} > \left(\frac{\mu_1+\mu_0}{2}\right)\right)$

$= P_{\mu_i}\left(\sqrt{n}(\bar{x}-\mu_i) > \sqrt{n}\left(\frac{\mu_1+\mu_0}{2} - \mu_i\right)\right)$

$= 1 - \Phi\left(\sqrt{n}\left(\frac{\mu_1+\mu_0}{2} - \mu_i\right)\right)$

$= \begin{cases} 1 - \Phi\left(\sqrt{n}\left(\frac{\mu_1-\mu_0}{2}\right)\right) & i = 0 \\ 1 - \Phi\left(\sqrt{n}\left(\frac{\mu_0-\mu_1}{2}\right)\right) & i = 1 \end{cases}$

$\to \begin{cases} 0 & i = 0 \\ 1 & i = 1 \end{cases}$

and using $\bar{x} \xrightarrow{SLLN} \mu_i$ when when μ_i is true so

$-2(\mu_1-\mu_0)\bar{x} - (\mu_1^2-\mu_0^2) \to \begin{cases} (\mu_1-\mu_0)^2 & \mu_0 \text{ true} \\ -(\mu_1-\mu_0)^2 & \mu_1 \text{ true} \end{cases}$

so $LR(\bar{x}) \to \begin{cases} \exp\{-\infty\} = 0 & \mu_0 \text{ true} \\ \exp\{\infty\} = \infty & \mu_1 \text{ true} \end{cases}$

So the $LR(\bar{x}) > 1$ makes the correct choice with certainty as $n \to \infty$.

(b) $BF(\mu_1 \mid \bar{x}) = \dfrac{\overline{\pi}(\{\mu_1\} \mid \bar{x})}{\overline{\pi}(\{\mu_0\} \mid \bar{x})} \Big/ \dfrac{1-p}{p}$

and $\overline{\pi}(\{\mu_1\} \mid \bar{x}) = \dfrac{(1-p)\exp\{-\frac{n}{2}(\bar{x}-\mu_1)^2\}}{p\exp\{-\frac{n}{2}(\bar{x}-\mu_0)^2\} + (1-p)\exp\{-\frac{n}{2}(\bar{x}-\mu_1)^2\}}$

$\overline{\pi}(\{\mu_0\} \mid \bar{x}) = \dfrac{p\exp\{-\frac{n}{2}(\bar{x}-\mu_0)^2\}}{p\exp\{-\frac{n}{2}(\bar{x}-\mu_0)^2\} + (1-p)\exp\{-\frac{n}{2}(\bar{x}-\mu_1)^2\}}$

Therefore $BF(\mu_1 \mid \bar{x}) = LR(\bar{x})$,

So the limiting values of $BF(\mu_1 \mid \bar{x})$ are the same as $LR(\bar{x})$

Now $RB(\{\mu_0\} \mid \bar{x}) = \dfrac{\overline{\pi}(\{\mu_0\} \mid \bar{x})}{\overline{\pi}(\{\mu_0\})}$

$= \dfrac{1}{p + (1-p)LR(\bar{x})}$ and so

$RB(\{\mu_0\} \mid \bar{x}) \to \begin{cases} 1/p & \text{when } \mu_0 \text{ is true} \\ 0 & \text{when } \mu_1 \text{ is true.} \end{cases}$

Since $BF(\mu_1 \mid \bar{x}) = LR(\bar{x})$ we have

$RB(\mu_0 \mid \bar{x}) = 1/(p + (1-p)BF(\mu_1 \mid \bar{x}))$

and $BF(\mu_0 \mid \bar{x}) = 1/BF(\mu_1 \mid \bar{x})$ then

$RB(\mu_0 \mid \bar{x}) = (p + (1-p)/BF(\mu_0 \mid \bar{x}))^{-1}$

and so if $BF(\mu_0|\bar{x}) > 1$ then $p + (1-p)/BF(\mu_0|\bar{x}) < p + (1-p) = 1$ and so $RB(\mu_0|\bar{x}) > 1$ and if $RB(\mu_0|\bar{x}) > 1$ then $BF(\mu_0|\bar{x}) > 1$.

Also $BF(\mu_0|\bar{x}) = \dfrac{(1-p)\,RB(\mu_0|\bar{x})}{1 - p\,RB(\mu_0|\bar{x})}$

and noting $(1-p)/(1 - p\,RB(\mu_0|\bar{x})) = 1/RB(\mu_1|\bar{x})$

we have $BF(\mu_0|\bar{x}) = \dfrac{RB(\mu_0|\bar{x})}{RB(\mu_1|\bar{x})}$.

(c) In class we showed k_0 is s.t.

$P_{\mu_0}(LR(\bar{x}) > k_0) = \alpha$ iff

$\log k_0 = \sqrt{n}(\mu_1 - \mu_0) z_{1-\alpha} - \frac{n}{2}(\mu_1 - \mu_0)^2$

So $k_0 < LR(\bar{x}) \le 1$ iff $\log k_0 \le \log LR(\bar{x}) \le 0$

iff $\log k_0 < n(\mu_1 - \mu_0)\bar{x} - \frac{n}{2}(\mu_1^2 - \mu_0^2) \le 0$

iff $\dfrac{\log k_0}{n(\mu_1 - \mu_0)} + \dfrac{\mu_1 + \mu_0}{2} < \bar{x} \le \dfrac{\mu_0 + \mu_0}{2}$

iff $\dfrac{\log k_0}{\sqrt{n}(\mu_1 - \mu_0)} + \sqrt{n}\dfrac{(\mu_1 - \mu_0)}{2} < \sqrt{n}(\bar{x} - \mu_0) \le \sqrt{n}\dfrac{(\mu_1 - \mu_0)}{2}$

iff $z_{1-\alpha} < \sqrt{n}(\bar{x} - \mu_0) \le \dfrac{\sqrt{n}(\mu_1 - \mu_0)}{2}$

Therefore, $P_{\mu_0}\left(k_0 < LR(\bar{x}) \le 1\right)$

$$= \begin{cases} \Phi\left(\sqrt{n}\,\frac{(\mu_1-\mu_0)}{2}\right) - (1-\alpha) & \text{when } \sqrt{n}\,\frac{(\mu_1-\mu_0)}{2} > z_{1-\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

$\to \alpha$ as $n \to \infty$ since $\mu_1 > \mu_0$.

Similarly $k_0 < LR(\bar{x}) \le 1$

iff $\frac{\log k_0}{\sqrt{n}(\mu_1-\mu_0)} - \sqrt{n}\,\frac{(\mu_1-\mu_0)}{2} < \sqrt{n}(\bar{x}-\mu_1) \le -\sqrt{n}\,\frac{(\mu_1-\mu_0)}{2}$

iff $z_{1-\alpha} - \sqrt{n}(\mu_1-\mu_0) < \sqrt{n}(\bar{x}-\mu_1) \le -\sqrt{n}\,\frac{(\mu_1-\mu_0)}{2}$

Therefore, $P_{\mu_1}\left(k_0 < LR(\bar{x}) \le 1\right)$

$$= \begin{cases} \Phi\left(-\sqrt{n}\,\frac{(\mu_1-\mu_0)}{2}\right) - \Phi\left(z_{1-\alpha} - \sqrt{n}(\mu_1-\mu_0)\right) & \\ \qquad \text{when } z_{1-\alpha} \le \sqrt{n}(\mu_1-\mu_0) & \\ 0 \qquad \text{otherwise.} & \end{cases}$$

$\to 0$ as $n \to \infty$.

The important point to note here is that for certain choices of α, n, μ_1, μ_0 we will have that $k_0 < 1$ and a positive probability that $k_0 < LR(\bar{x}) < 1$ and in such a case we will reject H_0 even though there is evidence in favor of H_0 as expressed by LR, BF and RB.

(d) $P_{\mu_0}\left(\sqrt{n}(\bar{X}-\mu_0) > \sqrt{n}(\bar{x}-\mu_0) \mid \bar{x}\right)$

$= 1 - \Phi(\sqrt{n}(\bar{x}-\mu_0))$ and since

$\sqrt{n}(\bar{X}-\mu_0) \overset{H_0}{\sim} N(0,1)$ this implies

$1-\Phi(\sqrt{n}(\bar{X}-\mu_0)) \sim U(0,1) \ \forall n$.

So $1-\Phi(\sqrt{n}(\bar{X}-\mu_0)) \xrightarrow{d} U(0,1)$

and the p-value does not converge to a single value when H_0 is true. But

when μ_0 is false $\bar{X} \xrightarrow{SLLN} \mu_1$ and so

the p-value $1-\Phi(\sqrt{n}(\bar{x}-\mu_0)) \to 1-\Phi(\infty) = 0$

(e) As mentioned already LR, BF and RB all characterize evidence whether for or against a particular hypothesis and seem to do so appropriately in this context. The hypothesis testing (Neyman-Pearson) approach can actually result in conclusions which contradict the evidence. This even happens asymptotically but, in any case, asymptotic good behavior is not a sufficient (only necessary) justification for a statistical procedure.

The p-value fails as a characterization of evidence because it does not indicate when there is evidence in favor (because of the uniform distribution under the null). This is just one of its defects and overall it is a poor basis for hypothesis assessment (instructor's opinion).

It isn't apparent from this example but the LR has difficulties as a characterization of evidence when there are more than two possibilities for a parameter. Both the BF and RB give evidence either for or against single values of the parameter without restriction. Typically, in continuous cases, the BF is defined by a mixture prior with a discrete mass at H_0. This leads to problems and there is no need to do this. If the BF is defined as a limit, as it should be, the limiting BF equals RB. In the instructor's opinion the RB is a much better way to deal with the concept of statistical evidence. There is more to this story yet. In particular discussing the strength of evidence, considering the question of choosing biased priors, etc.

In any case this simple example is a warning that there is still a lot to be worked out when considering a sound theory of statistical reasoning.

There was a typo in part (a).

(2) (a) We have that, putting $dv = (1-y)^{n-z-1}$, $u = y^z$

$$F(z) = \frac{n!}{z!\,(n-z-1)!}\left[-\frac{(1-y)^{n-z}}{n-z}\, y^z \Big|_\theta^1 + \frac{z}{n-z}\int_\theta^1 y^{z-1}(1-y)^{n-z}\,dy \right]$$

$$= \binom{n}{z}\theta^z(1-\theta)^{n-z} + \frac{n!}{(z-1)!\,(n-(z-1)-1)!}\int_\theta^1 y^{z-1}(1-y)^{n-(z-1)-1}\,dy$$

$$= \binom{n}{z}\theta^z(1-\theta)^{n-z} + \binom{n}{z-1}\theta^{z-1}(1-\theta)^{n-(z-1)} +$$

$$\frac{n!}{(z-2)!\,(n-(z-2)-1)!}\int_\theta^1 y^{z-2}(1-y)^{n-(z-2)-1}\,dy$$

etc. obtaining the result.

(b) Since $y = n\bar{x}$ is a mss we can base the test on $y \sim$ binomial(n,p). For $p_1 > p_0$ then

$$LR(\bar{x}) = \left(\frac{p_1}{p_0}\right)^y \left(\frac{1-p_1}{1-p_0}\right)^{n-y} = \left(\frac{p_1/(1-p_1)}{p_0/(1-p_0)}\right)^y \left(\frac{1-p_1}{1-p_0}\right)^n$$

and since $p_1 > p_0$ iff $p_1/(1-p_1) > p_0/(1-p_0)$ we have that the model has MLR form in y and it is strictly increasing in y. Therefore the UMP size α test is of the form

$$\phi(y) = \begin{cases} 1 & y > k_0 \\ \gamma & y = k_0 \\ 0 & y < k_0 \end{cases}$$

where k_0 satisfies $1 - F_{p_0}(k_0) \le \alpha < 1 - F_{p_0}(k_0 - 1)$
and $\gamma = \dfrac{\alpha - (1 - F_{p_0}(k_0))}{F_{p_0}(k_0) - F_{p_0}(k_0-1)}$.

3) The likelihood function is given by

$$L(\sigma^2 | x) = (2\pi\sigma^2)^{-n/2} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2\}$$

so by Factorization $\sum_{i=1}^{n}(x_i-\mu_0)^2$ is sufficient and since the max. of $L(\sigma^2|x)$ is attained at $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$ we can compute $\sum_{i=1}^{n}(x_i-\mu_0)^2$ from the LF so this statistic is a mss and we can base the test on $\sum_{i=1}^{n}(x_i-\mu_0)^2 \sim \sigma^2$ chi-squared(n). Now when $\sigma_1^2 > \sigma_0^2$ then

$$LR(\sum_{i=1}^{n}(x_i-\mu_0)^2) = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{-\frac{n}{2}} \exp\{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_{i=1}^{n}(x_i-\mu_0)^2\}$$

so the model is strictly MLR form in $\sum_{i=1}^{n}(x_i-\mu_0)^2$ since $1/\sigma_1^2 - 1/\sigma_0^2 < 0$.

Therefore the UMP size α test is of the form

$$\varphi\left(\sum_{i=1}^{n}(x_i-\mu_0)^2\right) = \begin{cases} 1 & \text{when } \frac{1}{\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu_0)^2 > \chi^2_{n,1-\alpha} \\ 0 & \text{otherwise} \end{cases}$$

since $\frac{1}{\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu_0)^2 \sim \chi^2_n$ when $\sigma^2 = \sigma_0^2$ and we can dispense with the boundary as $\sum_{i=1}^{n}(x_i-\mu_0)^2$ has a continuous distribution.

The power function is given by

$$\beta(\sigma^2) = P_{\sigma^2}\left(\frac{1}{\sigma_0^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 > \chi^2_{n,1-\alpha} \right)$$

$$= P_{\sigma^2}\left(\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 > \frac{\sigma_0^2}{\sigma^2} \chi^2_{n,1-\alpha} \right)$$

$= 1 - G_n\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{n,1-\alpha}\right)$ where G_n is the cdf of the chi-squared (n) distribution.