STA 3000 - 2020 Exercises 4 - Solutions

1. (a) $\beta(\theta) = E_\theta(\varphi_{\psi_0}) = E_\theta\left(\text{Prob}_U(\psi_0 \notin C_\gamma(u,x) \mid x)\right)$

$= \text{Prob}_\theta(\psi_0 \notin C_\gamma(u,x)) = 1 - \text{Prob}_\theta(\psi_0 \in C_\gamma(u,x))$

$\leq 1-\gamma$ when $\Psi(\theta)=\psi_0$ since $\text{Prob}_\theta(\Psi(\theta) \in C_\gamma(u,x)) \geq \gamma$

This shows that φ_{ψ_0} is size $1-\gamma$ for $H_0: \Psi(\theta)=\psi_0$

vs $H_a: \Psi(\theta) \neq \psi_0$.

(b) $\text{Prob}_\theta(\Psi(\theta) \in C_\gamma^*(u,x))$

$= \text{Prob}_\theta(\varphi_{\Psi(\theta)}(x) < u)$

$= E_\theta(\text{Prob}(\varphi_{\Psi(\theta)}(x) < u \mid x))$

$= E_\theta(1 - \varphi_{\Psi(\theta)}(x)) = 1 - E_\theta(\varphi_{\Psi(\theta)}(x))$

$= \text{Prob}_\theta(\Psi(\theta) \in C_\gamma(u,x))$

(c) Suppose C_γ a UMAU CR for Ψ. So

$1-\gamma \leq P_\theta(\Psi(\theta) \in C_\gamma(u,x))$ and $P_\theta(\Psi(\theta') \in C_\gamma(u,x)) \leq 1-\gamma$

$\forall \theta, \theta', \Psi(\theta) \neq \Psi(\theta')$. Thus $1 - E_\theta(\varphi_{\Psi(\theta)}) =$

$P_\theta(\Psi(\theta) \in C_\gamma(u,x)) \underset{\Psi(\theta)\neq\Psi(\theta')}{\geq} P_{\theta'}(\Psi(\theta) \in C_\gamma(u,x)) = 1 - E_{\theta'}(\varphi_{\Psi(\theta)})$

so $E_\theta(\varphi_{\Psi(\theta)}) \leq E_{\theta'}(\varphi_{\Psi(\theta)})$ $\forall \theta, \theta'$ $\Psi(\theta) \neq \Psi(\theta')$

so each $a_{\Psi(\theta)}$ is unbiased. Since C_γ minimizes $\text{Prob}_{\theta'}(\Psi(\theta) \in C_\gamma(u,x))$ uniformly in θ, θ', $\Psi(\theta) \neq \Psi(\theta')$ among all unbiased γ-CR's for Ψ and so $E_{\theta'}(a_{\Psi(\theta)})$ is maximized among all unbiased size $1-\gamma$ test fns. Basically the same argument shows the converse.

(d) If C_γ is nonrandomized then $C_\gamma(u,x) = C_\gamma(x)$ does not depend on u and so

$$a_\psi(x) = \text{Prob}(\psi \notin C_\gamma(u,x) \mid x)$$

$$= \text{Prob}(\psi \notin C_\gamma(x) \mid x) = \begin{cases} 1 & \psi \notin C_\gamma(x) \\ 0 & \text{otherwise}, \end{cases}$$

for every ψ. So a_ψ is not randomized so

$$C_\gamma^*(u,x) = \{\psi : a_\psi(x) < u\} = \{\psi : a_\psi(x) = 0\}$$

$$= \{\psi : \psi \in C_\gamma(x)\} = C_\gamma(x).$$

(e) Let $a_{1-\gamma}^*$ be the purely random size $1-\gamma$, i.e. $\varphi^*_{1-\gamma}(x) \equiv 1-\gamma$. Then putting

$\varphi_\psi = \varphi_{1-\gamma}$ $\forall \psi \in \Psi$ we have that

$$C(u,x) = \{\psi : \varphi_\psi(x) < u\}$$

$$= \begin{cases} \Psi & \text{when } 1-\gamma < u \\ \phi & \text{when } 1-\gamma \geq u. \end{cases}$$

Therefore $P(\psi \in C(u,x) \mid x) = P(u > 1-\gamma) = \gamma$

and so $P_\theta(\Psi(\theta) \in C(u,x)) = \gamma \quad \forall \theta$

and C is an exact γ-CR for $\psi = \Psi(\theta)$.

<u>Note</u> - the CR is either the full parameter space or the null set so it is absurd.

(2) The model for the sample is $\{f_\theta : \theta \in [0,1]\}$ where $f_\theta(x_1,\dots,x_n) = \theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})}$

$= \exp\{n\bar{x}\log\theta + n(1-\bar{x})\log(1-\theta)\}$

$= \exp\{n\bar{x}\log(\theta/(1-\theta)) + n\log(1-\theta)\}$

$= \exp\{n\bar{x}\,\eta(\theta) - n\log(1+\exp\{\eta(\theta)\})\}$

$= \exp\{n\bar{x}\,\eta - nA(\eta)\}$

so this is a 1-parameter exponential model. Now $\eta(\theta) = \log(\theta/(1-\theta))$ is a 1-1, increasing function of θ.

(a) By the above $H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$ is equivalent to testing $H_0: \eta = \eta(\theta_0)$ vs $H_a: \eta \neq \eta(\theta_0)$. By Theorem (5) proved in class the UMPU size α test is of the form

$$\varphi(x) = \begin{cases} 1 & n\bar{x} \notin (c_1, c_2) \\ \gamma_1 & n\bar{x} = c_1 \\ \gamma_2 & n\bar{x} = c_2 \\ 0 & n\bar{x} \in (c_1, c_2) \end{cases}$$

where $c_1, c_2, \gamma_1, \gamma_2$ are chosen to satisfy (i) $E_{\theta_0}(\varphi) = \alpha$ and (ii) $E_{\theta_0}(n\bar{x}\,\varphi(n\bar{x})) = \alpha E_{\theta_0}(n\bar{x})$
Under H_0 $n\bar{x} \sim$ binomial(n, θ_0) and so $E_{\theta_0}(n\bar{x}) = n\theta_0$ so we need to find $c_1, c_2, \gamma_1, \gamma_2$ satisfying

$$\text{(i)}\ \alpha = \sum_{k=0}^{c_1-1}\binom{n}{k}\theta_0^k(1-\theta_0)^{n-k} + \gamma_1\binom{n}{c_1}\theta_0^{c_1}(1-\theta_0)^{n-c_1}$$
$$+ \gamma_2\binom{n}{c_2}\theta_0^{c_2}(1-\theta_0)^{n-c_2} + \sum_{k=c_2+1}^{n}\binom{n}{k}\theta_0^k(1-\theta_0)^{n-c_1}$$

$$= F_{n,\theta_0}(c_1-1) + \gamma_1 f_{n,\theta_0}(c_1) + \gamma_2 f_{n,\theta_0}(c_2) + (1 - F_{n,\theta_0}(c_2))$$

where F_{n,θ_0}, f_{n,θ_0} are the cdf and pdf of binomial(n, θ_0)

$$\text{(ii)}\quad \alpha n\theta_0 = \sum_{k=0}^{c_1-1} k\binom{n}{k}\theta_0^k(1-\theta_0)^{n-k} + \gamma_1 c_1\binom{n}{c_1}\theta_0^{c_1}(1-\theta_0)^{n-c_1}$$

$$+ \gamma_2 c_2\binom{n}{c_2}\theta_0^{c_2}(1-\theta_0)^{n-c_2} + \sum_{k=c_2+1}^{n} k\binom{n}{k}\theta_0^k(1-\theta_0)^{n-k}$$

$$\therefore \alpha = \sum_{k=1}^{c_1-1}\binom{n-1}{k-1}\theta_0^{k-1}(1-\theta_0)^{n-k} + \gamma_1\binom{n-1}{c_1-1}\theta_0^{c_1-1}(1-\theta_0)^{n-c_1}$$

$$+ \gamma_2\binom{n-1}{c_2-1}\theta_0^{c_2-1}(1-\theta_0)^{n-c_2} + \sum_{k=c_2+1}^{n}\binom{n-1}{k-1}\theta_0^{k-1}(1-\theta_0)^{n-k}$$

$$= \sum_{k=0}^{c_1-2}\binom{n-1}{k}\theta_0^{k}(1-\theta_0)^{n-1-k} + \gamma_1\binom{n-1}{c_1-1}\theta_0^{c_1-1}(1-\theta_0)^{n-1-(c_1-1)}$$

$$+ \gamma_2\binom{n-1}{c_2-1}\theta_0^{c_2-1}(1-\theta_0)^{n-1-(c_2-1)} + \sum_{k=c_2}^{n-1}\binom{n-1}{k}\theta_0^{k}(1-\theta_0)^{n-1-k}$$

$$= F_{n-1,\theta_0}(c_1-2) + \gamma_1 f_{n-1,\theta_0}(c_1-1) + \gamma_2 f_{n-1,\theta_0}(c_2-1) + (1 - F_{n-1,\theta_0}(c_2-1))$$

So (using R functions) do a search to find all values (c_1, c_2) s.t.

$$\left.\begin{array}{l} F_{n,\theta_0}(c_1-1) + (1-F_{n,\theta_0}(c_2)) \le \alpha \le F_{n,\theta_0}(c_1) + (1-F_{n,\theta_0}(c_2-1)) \\ F_{n-1,\theta_0}(c_1-2) + (1-F_{n-1,\theta_0}(c_2-1)) \le \alpha \le F_{n-1,\theta_0}(c_1-1) + (1-F_{n-1,\theta_0}(c_2-2)) \end{array}\right\} (*)$$

then among those values (c_1, c_2) sat. $(*)$ find those values that have solutions $(\gamma_1, \gamma_2) \in [0,1]^2$ to the system of linear eq'ns

$$\begin{pmatrix} f_{n,\theta_0}(c_1) & f_{n,\theta_0}(c_2) \\ f_{n-1,\theta_0}(c_1-1) & f_{n-1,\theta_0}(c_2-1) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha - F_{n,\theta_0}(c_1-1) - (1-F_{n,\theta_0}(c_2)) \\ \alpha - F_{n-1,\theta_0}(c_1-2) - (1-F_{n-1,\theta_0}(c_2-1)) \end{pmatrix}$$

and there may be multiple solutions for given (c_1, c_2) as (γ_1, γ_2) are not unique.

(b) To determine a $(1-\alpha)$-confidence interval again we need to use R. For this construct a grid say of N equispaced θ values in $[0,1]$ say $\theta_1, \ldots, \theta_N$. For each θ_i use the algorithm in (a) to determine $c_1(\theta_i), c_2(\theta_i), \gamma_1(\theta_i), \gamma_2(\theta_i)$. Then for observed $n\bar{x}$ and u generated $u \sim U(0,1)$ the $(1-\alpha)$ confidence region is

$$C(u, n\bar{x}) = \{\theta : \alpha_\theta(n\bar{x}) < u\}$$

$$= \{\theta : c_1(\theta) < n\bar{x} < c_2(\theta)\} \cup \{\theta : n\bar{x} = c_1(\theta), \gamma_1(\theta) < u\}$$

$$\cup \{\theta : n\bar{x} = c_2(\theta), \gamma_2(\theta) < u\}$$

<u>Remark</u>. There are probably algorithms more efficient than that described here. The point, however, is that for one of the simplest and most basic problems <u>the</u> theory prescribes an interval which is nonintuitive because it is randomized and difficult to compute so nobody uses them. So the problem lies with the theory and we need to develop a better one rather than just ignoring the problem.

3. Clearly this is a 1-parameter exponential model and the mss is $n\bar{x} \sim \text{gamma}(n\eta, \theta)$

(a) Since $n\bar{x}$ has a continuous distribution $P_\theta(n\bar{x} = c) = 0$ $\forall c \in \mathbb{R}$. Therefore by Theorem ⓖ an unbiased test is the UMPU size α is of the form

$$\phi_{\theta_0}(x) = \begin{cases} 1 & n\bar{x} \notin (c_1, c_2) \\ 0 & \text{otherwise} \end{cases}$$

where c_1, c_2 are determined by

(i) $\alpha = E_{\theta_0}(\phi_{\theta_0}) = \int_0^{c_1} f_{n,\theta_0}(z)\,dz + \int_{c_2}^{\infty} f_{n,\theta_0}(z)\,dz$

$$= F_{n,\theta_0}(c_1) + (1 - F_{n,\theta_0}(c_2))$$

where $f_{n,\theta_0}(z) = \frac{\theta_0^{n\eta}}{\Gamma(n\eta)} z^{n\eta-1} e^{-\theta_0 z}$

with cdf F_{n,θ_0}

(ii) $\alpha E_{\theta_0}(n\bar{x}) = E_{\theta_0}(n\bar{x}\,\phi_{\theta_0}(n\bar{x}))$

$$= \int_0^{c_1} z f_{n,\theta_0}(z)\,dz + \int_{c_2}^{\infty} z f_{n,\theta_0}(z)\,dz$$

$$z f_{n,\theta_0}(z) = \frac{\Gamma(n\eta+1)}{\Gamma(n\eta)\theta_0} f_{n+1,\theta_0}(z) = \frac{n\eta}{\theta_0} f_{n+1,\theta_0}(z)$$

$$E_{\theta_0}(n\bar{x}) = \frac{n\eta}{\theta_0}$$

so, putting $r = n\lambda$,

$$\frac{\alpha r}{\theta_0} = \frac{r}{\theta_0}\left(F_{n+1,\theta_0}(c_1) + (1 - F_{n+1,\theta_0}(c_2))\right)$$

Therefore c_1, c_2 are determined by

$$\alpha = F_{n,\theta_0}(c_1) + (1-F_{n,\theta_0}(c_2)) = F_{n+1,\theta_0}(c_1) + (1-F_{n+1,\theta_0}(c_2))$$

or equivalently

$$1-\alpha = F_{n,\theta_0}(c_2) - F_{n,\theta_0}(c_1) = F_{n+1,\theta_0}(c_2) - F_{n+1,\theta_0}(c_1) \quad (*)$$

Now observe $F_{n,\theta_0}(x) = \int_0^x \frac{\theta_0^n}{\Gamma(r)} z^{r-1} e^{-\theta_0 z}\, dz$

(and putting $u = \theta_0 z$) $= \int_0^{\theta_0 x} \frac{u^{r-1}}{\Gamma(r)} e^{-u}\, du = F_{n,1}(\theta_0 x)$

and $F_{n+1,1}(x) = \int_0^x \frac{u^r}{\Gamma(r+1)} e^{-u}\, du$ (integration by parts $s = u^r,\ t = e^{-u}$)

$$= \frac{-u^r e^{-u}}{\Gamma(r+1)}\Big|_0^x + \frac{\Gamma(r)\, r}{\Gamma(r+1)} \int_0^x \frac{u^{r-1}}{\Gamma(r)} e^{-u}\, du$$

$$= \frac{x^r e^{-x}}{\Gamma(r+1)} + F_{n,1}(x).$$ So $(*)$ becomes

$$1-\alpha = F_{n,1}(\theta_0 c_2) - F_{n,1}(\theta_0 c_1)$$

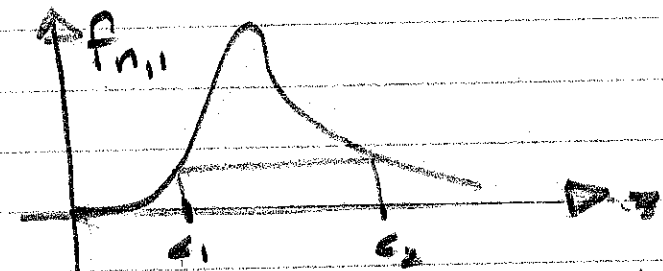
$$= \frac{(\theta_0 c_2)^n e^{-\theta_0 c_2}}{\Gamma(r+1)} - \frac{(\theta_0 c_1)^n e^{-\theta_0 c_1}}{\Gamma(r+1)} + F_{n,1}(\theta_0 c_2) - F_{n,1}(\theta_0 c_1)$$

and so (c_1, c_2) must satisfy $F_{n,1}(\theta_0 c_2) - F_{n,1}(\theta_0 c_1)$

$= 1-\alpha$ and $c_2^n e^{-\theta_0 c_2} = c_1^n e^{-\theta_0 c_1}$ (iff $f_{n,1}(c_1) = f_{n,1}(c_2)$).

Suppose that $\theta_0 = 1$ and so find $c_1(1), c_2(1)$ satisfying $F_{n,1}(c_2(1)) - F_{n,1}(c_1(1)) = 1-\alpha$ and

$c_1(1)^n e^{-c_1(1)} = c_2(1)^n e^{-c_2(1)}$



and these values can be solved for iteratively starting with $c_1 = 0$, $c_2 = \infty$ and increasing c_1 until the interval (c_1, c_2) contains $(1-\alpha)$ of the gamma$(n,1)$ probability. Then the UMPU size α test for $H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$ based on $n\bar{x}$ is equivalent to the test for $H_0: \theta = 1$ vs $H_a: \theta \neq 1$ based on $\theta_0 n\bar{x}$ which is (by theorem in the notes)

$$\varphi(\theta_0 n\bar{x}) = \begin{cases} 1 & \theta_0 n\bar{x} \notin (c_1(1), c_2(1)) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & n\bar{x} \notin (c_1(\theta_0), c_2(\theta_0)) \\ 0 & \text{otherwise} \end{cases}$$

where $c_i(\theta) = c_i(1)/\theta$

(b) The UMAU size α confidence region is given by $C(n\bar{x}) = \{\theta : c_1(\theta) \le n\bar{x} \le c_2(\theta)\}$

$$= \left\{\theta : \frac{c_1(1)}{\theta} \le n\bar{x} \le \frac{c_2(1)}{\theta}\right\}$$

$$= \left\{\theta : \frac{c_1(1)}{n\bar{x}} \le \theta \le \frac{c_2(1)}{n\bar{x}}\right\} = \left[\frac{c_1(1)}{n\bar{x}}, \frac{c_2(1)}{n\bar{x}}\right].$$

(11)

(4)

The location group $G = \{g \mid g \in \mathbb{R}\}$ acts on $\mathbb{R}^n$ via $T_g x = g1 + x$. If $X \sim N(\mu 1, \sigma^2 I)$ then $Y = T_g X \sim N_n((\mu+g)1, \sigma^2 I)$ and so the model is invariant. The action on $\Theta = \mathbb{R} \times \mathbb{R}^+$ is given by $\bar{T}_g(\mu, \sigma^2) = (g+\mu, \sigma^2)$. If $(\mu, \sigma^2) \in H_0$ then $\bar{T}_g(\mu, \sigma^2) \in H_0$ and similarly if $(\mu, \sigma^2) \in H_a$. Therefore G leaves the hypothesis testing problem invariant.

A minimal sufficient statistic for this problem is $(\bar{x}, s^2)$ and this is equivariant under G. So we can restrict our attention to invariant φ that depend on the data only through $(\bar{x}, s^2)$. Now G acts on $(\bar{x}, s^2)$ via $T_g^{**}(\bar{x}, s^2) = (\bar{x}+g, s^2)$ and so s^2 is a maximal invariant and any invariant φ depends on $(\bar{x}, s^2)$ only through $s^2 = \|x - \bar{x}1\|^2/(n-1)$.

When $x_1, \ldots, x_n \sim N(\mu, \sigma^2)$ then

$$y = \frac{(n-1)s^2}{\sigma^2} \sim \text{Chisquare}(n-1)$$

and so $y = (n-1)s^2$ density (proportional to)

$$\left(\frac{y}{\sigma^2}\right)^{\frac{n-1}{2}-1} e^{-y/2\sigma^2} \frac{1}{\sigma^2}$$

Now consider the hypothesis problem $H_0: \sigma^2 = \sigma_0^2$ vs $H_a: \sigma^2 = \sigma_1^2$ where $\sigma_1^2 > \sigma_0^2$. The Fundamental Lemma says that the MP size α test of H_0 vs H_a, based on y, is of the form

$$\phi(y) = \begin{cases} 1 & \left(\frac{\sigma_0^2}{\sigma_1^2}\right) e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)y} > k \\ \gamma & = k \\ 0 & < k \end{cases}$$

for some γ, k to get size α.
This is equivalent to (since $\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2} < 0$)

$$\phi(y) = \begin{cases} 1 & y > k' \\ \gamma & = k' \\ 0 & < k' \end{cases}$$

for some k'. Since $y \sim \sigma_0^2$ Chisquare$(n-1)$ under H_0 we can choose $k' = \sigma_0^2 \chi^2_\alpha(n-1)$ to get exact size α. Therefore

$$\phi(y) = \begin{cases} 1 & (n-1)s^2 > \sigma_0^2 \chi^2_\alpha(n-1) \\ 0 & < \end{cases}$$

is MP invariant size α for $H_0: \sigma^2 = \sigma_0^2$ vs $H_a: \sigma^2 = \sigma_1^2$. Because it doesn't depend on σ_1^2 it is UMP invariant size α for $H_0: \sigma^2 = \sigma_0^2$ vs $H_a: \sigma^2 > \sigma_0^2$.
Finally if $\sigma^2 < \sigma_0^2$ then

$$P\left((n-1)s^2 > \sigma_0^2 \chi^2_\alpha(n-1)\right) = P\left(\frac{(n-1)s^2}{\sigma^2} > \frac{\sigma_0^2}{\sigma^2}\chi^2_\alpha(n-1)\right)$$

$< \alpha$ since $\sigma_0^2/\sigma^2 > 1$. Therefore ϕ is size α for $H_0: \sigma^2 \le \sigma_0^2$ vs $H_a: \sigma^2 > \sigma_0^2$ and so is UMP invariant size α for this problem!

[redacted] 5

(a) (i) $$\left(\left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right]\left[\begin{pmatrix} a_1' & a_2' \\ 0 & b'\end{pmatrix}; c'\right]\right)\left[\begin{pmatrix} a_1'' & a_2'' \\ 0 & b''\end{pmatrix}; c''\right]$$

$$= \left[\begin{pmatrix} a_1a_1' & a_1a_2'+a_2b' \\ 0 & bb'\end{pmatrix}; cc'\right]\left[\begin{pmatrix} a_1'' & a_2'' \\ 0 & b''\end{pmatrix}; c''\right]$$

$$= \left[\begin{pmatrix} a_1a_1'a_1'' & a_1a_1'a_2''+a_1a_2'b''+a_2b'b'' \\ 0 & bb'b''\end{pmatrix}; cc'c''\right]$$

$$= \left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right]\left(\left[\begin{pmatrix} a_1' & a_2' \\ 0 & b'\end{pmatrix}; c'\right]\left[\begin{pmatrix} a_1'' & a_2'' \\ 0 & b''\end{pmatrix}; c''\right]\right),$$

(ii) $$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}; 1\right]\left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right] = \left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right]$$

$$= \left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right]\left[\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}; 1\right]$$

(iii) $$\left[\begin{pmatrix} \frac{1}{a_1} & -\frac{a_2}{a_1 b} \\ 0 & \frac{1}{b}\end{pmatrix}; \frac{1}{c}\right]\left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right]$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}; 1\right]$$

$$= \left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right]\left[\begin{pmatrix} \frac{1}{a_1} & -\frac{a_2}{a_1 b} \\ 0 & \frac{1}{b}\end{pmatrix}; \frac{1}{c}\right]$$

$\therefore$ G_1 is a group

(b) (i) $$T_{\left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right]}\, T_{\left[\begin{pmatrix} a_1' & a_2' \\ 0 & b'\end{pmatrix}; c'\right]} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}\begin{pmatrix} a_1' & a_2' \\ 0 & b'\end{pmatrix} x \\ c c' \begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}\begin{pmatrix} a_1' & a_2' \\ 0 & b'\end{pmatrix} y \end{pmatrix} = T_{\left[\begin{pmatrix} a_1 & a_2 \\ 0 & b\end{pmatrix}; c\right]\left[\begin{pmatrix} a_1' & a_2' \\ 0 & b'\end{pmatrix}; c'\right]} \begin{pmatrix} x \\ y \end{pmatrix}$$

(ii) $$T_{\left[\begin{pmatrix}1&0\\0&1\end{pmatrix};1\right]}\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix} = \begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix}$$

$\therefore$ this is an action of G_1 on $\mathbb{R}^4$

$$\text{Now } T_{\left[\begin{pmatrix}a_1&a_2\\0&b\end{pmatrix};c\right]}\begin{pmatrix}\underline{x}_1\\ \underline{x}_2\end{pmatrix} = T_{\left[\begin{pmatrix}a_1'&a_2'\\0&b'\end{pmatrix};c'\right]}\begin{pmatrix}\underline{x}_1\\ \underline{x}_2\end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix}a_1&a_2\\0&b\end{pmatrix}\underline{x}_1 = \begin{pmatrix}a_1'&a_2'\\0&b'\end{pmatrix}\underline{x}_1$$

$$c\begin{pmatrix}a_1&a_2\\0&b\end{pmatrix}\underline{x}_2 = c'\begin{pmatrix}a_1'&a_2'\\0&b'\end{pmatrix}\underline{x}_2$$

$$\Leftrightarrow \quad X = \begin{pmatrix}\underline{x}_1 & \underline{x}_2\end{pmatrix} = \begin{pmatrix}a_1&a_2\\0&b\end{pmatrix}^{-1}\begin{pmatrix}a_1'&a_2'\\0&b'\end{pmatrix} X \begin{pmatrix}1&0\\0&\frac{c'}{c}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{a_1} & -\frac{a_2}{a_1 b}\\ 0 & \frac{1}{b}\end{pmatrix}\begin{pmatrix}a_1'&a_2'\\0&b'\end{pmatrix} X \begin{pmatrix}1&0\\0&\frac{c'}{c}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{a_1'}{a_1} & \frac{a_2'}{a_1} - \frac{a_2}{a_1}\frac{b'}{b}\\ 0 & \frac{b'}{b}\end{pmatrix} X \begin{pmatrix}1&0\\0&\frac{c'}{c}\end{pmatrix}$$

$$= \begin{pmatrix}A&B\\0&C\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}\begin{pmatrix}1&0\\0&D\end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix} = \begin{pmatrix}Ax_{11}+Bx_{21} & Ax_{12}+Bx_{22}\\ Cx_{21} & Cx_{22}\end{pmatrix}\begin{pmatrix}1&0\\0&D\end{pmatrix}$$

$$= \begin{pmatrix}Ax_{11}+Bx_{21} & D(Ax_{12}+Bx_{22})\\ Cx_{21} & CDx_{22}\end{pmatrix}$$

$$\Leftrightarrow \quad (A-1)x_{11} + Bx_{21} = 0 \quad ①$$
$$(AD-1)x_{12} + BDx_{22} = 0 \quad ②$$
$$(C-1)x_{21} = 0 \quad ③$$
$$(CD-1)x_{22} = 0 \quad ④$$

Now clearly ①-④ are satisfied if $X = 0$ so the action is not free. But note that

$$P(\det X \neq 0,\ x_{21} \neq 0,\ x_{22} \neq 0) = 1$$

So after deleting $\{\binom{x_1}{x_2} \mid \det(x_1\ x_2) = 0 \text{ or } x_{21} = 0 \text{ or } x_{22} = 0\}$ from $\mathbb{R}^4$ we have that

③ implies $C = 1$ or $b' = b$ and then ④ implies $D = 1$ or $c' = c$ and ① and ② become

$$\begin{pmatrix} A-1 & B \\ A-1 & B \end{pmatrix} X' = 0 \Leftrightarrow \begin{pmatrix} A-1 & B \\ A-1 & B \end{pmatrix} = 0$$

$\Leftrightarrow A = 1$ or $a_1 = a_1'$ and $B = 0$ or $a_2' = a_2$

which implies that the action is free on this reduced sample space (assumed hereafter).

(c) If $z \sim N_2(0, \Sigma)$ stat. ind. of $y \sim N_2(0, \Delta\Sigma)$

then $T_{[\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right); d]} \binom{z}{y} = \begin{pmatrix} \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) z \\ d \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) y \end{pmatrix}$

$$\sim N_2\left(\underset{\sim}{0}, \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}'\right) * N_2\left(\underset{\sim}{0}, d^2 \Delta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}'\right)$$

and so the model is invariant.

We have that $\bar{T}_{\left[\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}; d\right]}\begin{pmatrix}\Sigma \\ \Delta\end{pmatrix} = \begin{pmatrix}\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}\Sigma\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}' \\ d^2\Delta\end{pmatrix}$

(d) $\Phi\left(\bar{T}_{\left[\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}; d\right]}\begin{pmatrix}\Sigma \\ \Delta\end{pmatrix}\right) = d^2\Delta$ and so

$\Phi(\theta_1) = \Phi(\theta_2)$ implies $\Delta_1 = \Delta_2$ which implies $\Phi(\bar{T}_g\theta_1) = \Phi(\bar{T}_g\theta_2)$.

$\boxed{\text{change } A \text{ to } \Phi}$

Therefore Φ is equivariant.

The action of G_1 on the action space is given by $T_g^* A(\theta) = \Phi(\bar{T}_g\theta) = d^2\Phi(\theta)$

Therefore $\tilde{T}_g\Delta = d^2\Delta$

We then have that $L(\bar{T}_g\theta, T_g^* a) = \rho\left(\frac{d^2 a}{d^2\Delta}\right) = L(\theta, a)$ and the decision problem is invariant under G_1.

Let δ be a decision f^n s.t. $\int_0^\infty |a|\delta(x, y, da) < \infty$, and put $d^*(x, y) = \int_0^\infty a\,\delta(x, y, da)$. Then

$R_\delta(\theta) = \int_{\mathbb{R}^4}\int_0^\infty \rho\left(\frac{a}{\Delta}\right)\delta(x, y, da)\,P_\theta(dx\,dy)$

assume ρ convex

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$\geq \int_{\mathbb{R}^4} \rho\left(\frac{d(x, y)}{\Delta}\right) P_\theta(dx\,dy)$

and so we can restrict to nonrandomized rules.

(e) If $d^*(\underset{\sim}{x}, \underset{\sim}{y}) = k_1 y_2^2 / x_2^2$ then

$$d^*\left(T_g\begin{pmatrix} x \\ y \end{pmatrix}\right) = d^*\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \underset{\sim}{x}, d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \underset{\sim}{y}\right)$$

$$= k_1 \frac{d^2 c^2 y_2^2}{c^2 x_2^2} = d^2 k_1 \frac{y_2^2}{x_2^2}$$ and d^* is equivariant.

Now suppose $d^*(\underset{\sim}{x}, \underset{\sim}{y})$ is equivariant.

Then $d^*\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \underset{\sim}{x}, d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \underset{\sim}{y}\right) = d^2 d^*(\underset{\sim}{x}, \underset{\sim}{y})$

Now for $(\underset{\sim}{x}, \underset{\sim}{y})$ in the restricted sample space we have that $x_2, y_2 \neq 0$ and $(\underset{\sim}{x}, \underset{\sim}{y})$ is invertible. Therefore we can choose a, b s.t. $(a\ b)(\underset{\sim}{x}\ \underset{\sim}{y}) = (1\ 1)$. Now choose $c = 1/x_2$ and $d = x_2/y_2$. Then we have

$$d^*\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \frac{x_2^2}{y_2^2} d^*(\underset{\sim}{x}, \underset{\sim}{y})$$

Therefore $d^*(\underset{\sim}{x}, \underset{\sim}{y}) = k_1 \frac{y_2^2}{x_2^2}$ where

$k_1 = d^*\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.

Now let $g = [\Sigma_{LT}^{-1}, \Delta^{-\frac{1}{2}}]$ where Σ_{LT} is the Cholesky factor of Σ. Then

$$R_{d_*}(\theta) = R_{d_*}(T_g\theta) = R_{d_*}(I,1)$$
$$= \int_{\mathbb{R}^4} \rho\left(\frac{k_1 y_2^2}{x_3^2}\right) P_{(I,1)}(dx, dy)$$

and the optimal estimator is obtained by obtaining the k_1 which minimizes the above integral.

(f) Rs in (a)-(e).

Note This problem is invariant under two different groups and they each lead to different optimal invariant procedures — a big problem for invariance in decision theory!

6. (a) (i) $(g_1 \cdot g_2)\cdot g_3 = (g_1+g_2)+g_3 = g_1+(g_2+g_3)$
$= g_1\cdot(g_2\cdot g_3)$
(ii) $\underset{\sim}{0}\cdot g = 0+g = g+\underset{\sim}{0} = g = g\cdot \underset{\sim}{0}$ so $\underset{\sim}{0}$
$\underset{\sim}{0}$ is the identity
(iii) $g\cdot(-g) = g+(-g) = g-g = \underset{\sim}{0}$
and so $g^{-1} = -g$
Therefore G is a group.

(b) Suppose $T_g X_1 = T_g X_2$ then $X_1 + g\underset{\sim}{1}' = X_2 + g\underset{\sim}{1}'$
so we must have $X_1 = X_2$ and so T_g is 1-1.
Also $T_{g_1 \cdot g_2} X = X + (g_1\cdot g_2)\underset{\sim}{1}' = X+(g_1+g_2)\underset{\sim}{1}'$
$= X + g_1' + g_2\underset{\sim}{1}' = T_{g_1}(X+g_2\underset{\sim}{1}') = T_{g_1}T_{g_2}X.$
Finally $T_0 X = X + \underset{\sim}{0}\underset{\sim}{1}' = X$. Therefore
G acts on the sample space
We have $T_{g_1}X_1 = T_{g_2}X$ iff $X+g_1\underset{\sim}{1}' =$
$X_2 + g_2\underset{\sim}{1}'$ iff $g_1 = g_2$ and so the action
is free.

(c) If $\underset{\sim}{x} \sim N_k(\underset{\sim}{\mu}, I)$ then $\underset{\sim}{y} = T_g \underset{\sim}{x} \sim$
$N_k(g+\underset{\sim}{\mu}, I)$ and so $T_g X \overset{\text{iid}}{\sim} N(g+\underset{\sim}{\mu}, I)$
which is in the model. So G leaves the
model invariant.
Clearly $\bar{T}_g \Theta = g + \Theta$ and since $\Psi = \Theta$
the action is the same on Ψ. If $\psi_1, \psi_2 \in \Psi$
then putting $g = \psi_2 - \psi_1$ we have $\bar{T}_g \psi_1$
$= \psi_1 + \psi_2 - \psi_1 = \psi_2$ so G acts transitively
on Ψ
Finally $L(\bar{T}_g\theta, T_g^{\psi}\psi) = \|g+\theta-(g+\psi)\|^2$
$= \|\theta - \psi\|^2$ and so the decision problem is
invariant under G.

(d) The density of $\underset{\sim}{X}$ is

$$(2\pi)^{-nk/2} \exp\{-\frac{n}{2}\|\bar{\underset{\sim}{x}}-\theta\|^2 - \frac{1}{2}\sum_{i=1}^{n}(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})'(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})\}$$

so we have that $\bar{x}$ is sufficient by Factorization and since the MLE of θ is $\bar{X}$ (so we can obtain $\bar{X}$ from the likelihood) this proves $\bar{X}$ is a m.s.s.

Now $\overline{T_g x} = \frac{1}{n}\sum_{i=1}^{n} T_g x_i = \frac{1}{n}\sum_{i=1}^{n}(x_i+g) = \bar{\underset{\sim}{x}}+g$ so $\bar{\underset{\sim}{x}}$ is equivariant. Since the problem is convex and $\bar{\underset{\sim}{x}}$ is equivariant the optimal invariant estimator is a function of $\bar{\underset{\sim}{x}}$.

(e) Since we can reduce the data to $\bar{\underset{\sim}{x}}$ and the group acts on $\bar{x}$ via $T_g\bar{x} = \bar{\underset{\sim}{x}}+g$ we can treat the problem as if the data is $\bar{\underset{\sim}{x}}$. Note that a transformation variable for this problem is $[\bar{\underset{\sim}{x}}] = \bar{\underset{\sim}{x}}$ and so $D(\bar{\underset{\sim}{y}}) = T^{-1}_{[\bar{x}]}\bar{\underset{\sim}{y}} = \bar{\underset{\sim}{y}}-\bar{\underset{\sim}{x}} = \underset{\sim}{0}$ and the max. invariant is a constant. Thus $\bar{\underset{\sim}{x}} \sim N_k(\theta, \frac{1}{n}I)$ is stat. ind. of $D(\bar{\underset{\sim}{x}})$ and the Pitman estimate minimizes

$$\int_{\mathbb{R}^k} \|\theta - d(\bar{\underset{\sim}{x}})\|^2 f_\theta(\bar{\underset{\sim}{x}})\, d\bar{\underset{\sim}{x}}$$

and writing $\bar{\underset{\sim}{x}} = \theta + \underset{\sim}{z}$ with $\underset{\sim}{z} \sim N_k(0, \frac{1}{n}I)$

$$= \int_{\mathbb{R}^k} \|d(\underset{\sim}{z})\|^2 f_{\underset{\sim}{0}}(\underset{\sim}{z})\, d\underset{\sim}{z}$$

$$= \int_{\mathbb{R}^k} \|\underset{\sim}{z} + d(\underset{\sim}{0})\|^2 f_0(\underset{\sim}{z})\, d\underset{\sim}{z}$$ which by

least-squares is minimized by $d(\underset{\sim}{0}) = E(\underset{\sim}{z}) = \underset{\sim}{0}$

and so $d(\underline{x}) = \underline{x}$ is the optimal invariant estimator.

(f) Suppose, $\|\theta_1\|^2 = \|\theta_2\|^2$

Then $\|\bar{T}_g\theta_1\|^2 = \sum_{i=1}^{k}(g_i+\theta_{1i})^2$

$= \|g\|^2 + 2\sum_{i=1}^{k} g_i\theta_{1i} + \|\theta_1\|^2$

and this equals $\|\theta_2\|^2$ iff $\sum_{i=1}^{n} g_i\theta_{1i}$

$= \sum_{i=1}^{n} g_i\theta_{2i}$ and this need not be true.

For example, suppose $k=2$, $g = \binom{1}{1}$

$\theta_1 = \binom{1}{0}$, $\theta_2 = \binom{1/\sqrt{2}}{-1/\sqrt{2}}$. Then $\|\theta_1\|^2 = \|\theta_2\|^2$

but $\sum_{i=1}^{2} g_i\theta_{1i} = 1$ while $\sum_{i=1}^{2} g_i\theta_{2i} = 0$

Therefore this problem will not be invariant under G.

<u>Remark</u> This is a general problem for invariance in decision theory as only certain $\mathcal{I}$ are amenable to the approach.