The Measurement of Statistical Evidence
Lecture 5 - part 2

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**Example** location normal

- \( x = (x_1, \ldots, x_n) \sim N(\mu, \sigma_0^2) \) with \( \mu \in \mathbb{R}^1, \sigma_0^2 \) known and \( \pi \) a \( N(\mu_0, \tau_0^2) \) dist., recall we discussed eliciting \((\mu_0, \tau_0^2)\) and derived posterior

\[
\mu | x \sim N(\mu_x, \tau_x^2)
\]

\[
\mu_x = \tau_x^2 \left( \frac{\mu_0}{\tau_0^2} + \frac{n \bar{x}}{\sigma_0^2} \right), \quad \tau_x^2 = \left( \frac{1}{\tau_0^2} + \frac{n}{\sigma_0^2} \right)^{-1}
\]

- since \( \bar{x} \) is a mss \( RB(\mu | x) = RB(\mu | \bar{x}) \)

\[
RB(\mu | \bar{x}) = \frac{f_{\mu, \bar{x}}(\bar{x})}{m_{\bar{x}}(\bar{x})} = \frac{(n/2\pi\sigma_0^2)^{1/2} \exp\left\{-n(\bar{x} - \mu)^2 / 2\sigma_0^2\right\}}{m_{\bar{x}}(\bar{x})}
\]

where with \( Z_1, Z_2 \sim N(0, 1) \) the prior predictive dist. of \( \bar{X} \)

\[
\bar{X} = \mu + \frac{\sigma_0}{\sqrt{n}} Z_1 = \mu_0 + \tau_0 Z_2 + \frac{\sigma_0}{\sqrt{n}} Z_1 \sim N(\mu_0, \tau_0^2 + \sigma_0^2 / n)
\]

\[
m_{\bar{x}}(\bar{x}) = \left( 2\pi(\tau_0^2 + \sigma_0^2 / n) \right)^{-1/2} \exp\left\{- (\bar{x} - \mu_0)^2 / 2(\tau_0^2 + \sigma_0^2 / n) \right\}
\]
- so the relative belief estimate is \( \mu(x) = \bar{x} \rightarrow \mu_{true} \) as \( n \rightarrow \infty \) and

\[
P_l(x) = \{ \mu : (n/2\pi\sigma_0^2)^{1/2} \exp\{ -n(\bar{x} - \mu)^2 / 2\sigma_0^2 \} > m_{\bar{x}}(\bar{x}) \}
\]

\[
= \{ \mu : n(\bar{x} - \mu)^2 / 2\sigma_0^2 < - \log \left[ (n/2\pi\sigma_0^2)^{-1/2} m_{\bar{x}}(\bar{x}) \right] \}
\]

\[
= \bar{x} \pm \sqrt{\frac{2\sigma_0^2}{n}} \log \left[ \frac{(n/2\pi\sigma_0^2)^{1/2}}{m_{\bar{x}}(\bar{x})} \right] = \bar{x} \pm c(\bar{x})
\]

- so for accuracy assessment quote half-length \( c(\bar{x}) \) and posterior content

\[
\Pi(P_l(x) \mid \bar{x}) = \Phi \left( \frac{\bar{x} + c(\bar{x}) - \mu_x}{\tau_x} \right) - \Phi \left( \frac{\bar{x} - c(\bar{x}) - \mu_x}{\tau_x} \right)
\]

- note

\[
c^2(\bar{x}) = \frac{2\sigma_0^2}{n} \left\{ \frac{1}{2} \log \frac{n}{2\pi\sigma_0^2} - \log m_{\bar{x}}(\bar{x}) \right\}
\]

\[
= \frac{\sigma_0^2}{n} \left\{ \log \frac{\tau_0^2 + \sigma_0^2 / n}{\sigma_0^2 / n} + \frac{(\bar{x} - \mu_0)^2}{\tau_0^2 + \sigma_0^2 / n} \right\}
\]

and so \( P_l(x) \neq \phi \) and \( P_l(x) \rightarrow \{ \mu_{true} \} \)
- for any $\gamma \leq \Pi(Pl(x) \mid \bar{x})$ the $\gamma$-relative belief region $C_\gamma(x) = \bar{x} \pm k_\gamma(\bar{x})$ can be quoted where $k_\gamma(\bar{x})$ satisfies

$$\Phi\left(\frac{\bar{x} + k_\gamma(\bar{x}) - \mu_x}{\tau_x}\right) - \Phi\left(\frac{\bar{x} - k_\gamma(\bar{x}) - \mu_x}{\tau_x}\right) = \gamma$$

which can be obtained via simple tabulation

- to assess $H_0 := \{\mu_*\}$ compute $RB(\mu_* \mid \bar{x})$ and the strength

$$\Pi(RB(\mu \mid \bar{x}) \leq RB(\mu_* \mid \bar{x}) \mid \bar{x})$$

$$= \Pi\left((\bar{x} - \mu)^2 \geq -\frac{2\sigma^2_0}{n} \log[(n/2\pi\sigma^2_0)^{-1/2} m_{\bar{x}}(\bar{x}) RB(\mu_* \mid \bar{x}) \mid \bar{x}] \mid \bar{x}\right)$$

$$= \Pi\left((\bar{x} - \mu)^2 \geq d^2(\mu_*) \mid \bar{x}\right)$$

$$= \Pi\left((-\infty, \bar{x} - d(\mu_*)) \cup (\bar{x} + d(\mu_*), \infty) \mid \bar{x}\right)$$

$$= \Phi\left(\frac{\bar{x} - d(\mu_*) - \mu_x}{\tau_x}\right) + 1 - \Phi\left(\frac{\bar{x} + d(\mu_*) - \mu_x}{\tau_x}\right)$$
- while exact formulas for these quantities can be convenient these are generally not available and we need to proceed numerically

- so we do this here for this problem with a specific example (discussed previously)

- the elicitation (based on the mean lying in \((l, u) = (3, 10)\) with prob. 0.99) resulted in \((\mu_0, \tau_0) = (6.5, 1.36)\)

- if \(\sigma_0^2 = 2\) and \(n = 10\), \(\bar{x} = 7.3\) is observed, then the posterior of \(\mu\) is \(N(\mu_x, \tau_x^2) = N(7.23, 0.18)\)

- suppose interest is in \(\psi = \Psi(\mu) = \mu^3\) and a meaningful difference from \(\mu^3_{true}\) is \(\delta = 10\)

- so create grid of 60 points \(\text{grid} = \{10, 30, 50, \ldots, 1190\}\) and for \(i \in \text{grid}\) we estimate the prior and posterior contents of \(i \pm 10\) by simulating from the prior and posterior of \(\mu^3\) and estimate the densities over these intervals by dividing these estimates by \(2\delta\) giving us density histograms
Program in R

```r
# set up
sigma0=sqrt(2)
mu0=6.5
tau0=sqrt(1.36)
n=10
xbar=7.3

\[
\text{taux2} = \frac{1}{\frac{1}{\text{tau0}^2} + \frac{n}{\text{sigma0}^2}}
\]

\[
\text{mux} = \text{taux2} \left( \frac{\text{mu0}}{\text{tau0}^2} + \frac{n \times \text{xbar}}{\text{sigma0}^2} \right)
\]

\[
\text{taux} = \sqrt{\text{taux2}}
\]

numgrid=60

grid=-10+20*c(1:numgrid)
```

# prior calculations
priorprob=0*grid
nmonte=500000
mugen=rnorm(nmonte, mu0, tau0)
mugen3=mugen**3
for (i in 1:100) {
  for (j in 1:500000) {
    if (mugen3[j] > grid[i]-10 & mugen3[j] <= grid[i]+10 ){
      priorprob[i]=priorprob[i]+1
    }
  }
}
priorprob=priorprob/nmonte
priordens=priorprob/20
plot(grid, priordens, xlab="mu cubed", ylab="prior density", type="l", lty=2)
# posterior calculations
postprob=0*grid
nmonte=500000
mugen=rnorm(nmonte,mux,taux)
mugen3=mugen**3
for (i in 1:numgrid){
    for (j in 1:nmonte) {
        if (mugen3[j] > grid[i]-10 & mugen3[j] <= grid[i]+10 ){
            postprob[i]=postprob[i]+1
        }
    }
}
postprob=postprob/nmonte
postdens=postprob/20
plot(grid,postdens,xlab="mu^3", ylab=expression("prior - - - and posterior __ __ "), type="l",lty=1)
lines(grid,priordens,type="l",lty=2)
# compute RB
RB = postdens/priordens
plot(grid, RB, xlab="mu^3", ylab="RB", type="l", lty=1)
# find \(\mu^3(x)\)

\[\text{imax}=1\]

\[\text{RBmax}=\text{RB[imax]}\]

\[
\text{for (i in 1:numgrid)}{\text{if( RB[i]>RBmax)}{i}\text{imax=i \text{RBmax=RB[i]}}}
\]

\[
\text{cat("\mu^3(x)= ",grid[imax],"RB(\mu^3(x) \mid x) = ",RB[imax],"\n")}
\]

\[
\mu^3(x)= 390 \text{ RB(}390\mid x\text{)} = 3.719956
\]

- so \(\mu^3(x) = 390, RB(390 \mid x) = 3.719956\) and note \(\bar{x}^3 = 7.3^3 = 389.017\)

so good approximation
# approximate Pl(x) and its posterior content

plauscont=0

for (i in 1:numgrid){
    if( RB[i]>1){
        cat(grid[i]," ")
        plauscont=plauscont+postprob[i]
    }
}

plauscont

290 310 330 350 370 390 410 430 450 470 490

[1] 0.903396

- so $PL(x) = (280, 500)$ with posterior content 0.903396
- to get a 0.80-relative belief region

```r
k=2
cont=0
for (i in 1:numgrid){
  if( RB[i] > k){
    cat(grid[i], " ")
    cont=cont+postprob[i]  }
}
cont
330 350 370 390 410 430 450 > cont
[1] 0.697656
- too low
```
k=1.5
cont=0
for (i in 1:numgrid){
  if( RB[i] > k){
    cat(grid[i]," ")
    cont=cont+postprob[i]
  }
}
cont
310 330 350 370 390 410 430 450 470 > cont
[1] 0.820932
- good enough so $C_{0.82}(x) = (290, 470)$
- assess $H_0 : \mu^3 = 625$ or equivalently, since $625 \in (630 - 10, 630 + 10)$ where 630 is a grid point $H_0 = (620, 640)$ and $RB(630 | x)$  

# assess H_0: mu^3=625 by getting RB at nearest gridpoint, here 630  
for (i in 1:numgrid){  
cat(i,grid[i],RB[i],"\n")  
}  
- obtain $RB(630 | x) = 0.05116493$ so evidence against and strength is  

#strength  
strength=0  
for (i in 1:numgrid){  
if (RB[i]<=RB[32]){  
strength=strength+postprob[i]  
}  
}  
strength  
[1] 0.003552  
- so strong evidence against $H_0$

- \( x = (x_1, \ldots, x_m) \stackrel{i.i.d.}{\sim} N(\mu, \sigma_0^2) \) ind. of \( y = (y_1, \ldots, y_n) \stackrel{i.i.d.}{\sim} N(\nu, \sigma_0^2) \) and \( \psi = \Psi(\mu, \nu) = \mu / \nu \)

- give an elicitation algorithm for \( \mu \) and \( \nu \) that takes into account that we know something about \( \psi \)

- it is probably okay to allow 0 as a possible (probable) value for \( \mu \) but not both \( \mu \) and \( \nu \) as that would suggest the possibility that \( \psi \) is not defined

- need to choose a relevant \( \delta \)

- then proceed as we did in the previous example for problems \( \mathbf{E} \) and \( \mathbf{H} \)

- maybe take \( m = n = 10, \sigma_0^2 = 1 \) and suppose \( \mu_{true} = 2\nu_{true} \) where \( \nu_{true} = 10 \) and generate the samples \( x \) and \( y \)