(1) Suppose we have a population Ω and for each $\omega \in \Omega$ we measure $(X(\omega), Y(\omega))$ where $X(\omega) \in \{1, 2, 3\}$ and $Y(\omega) \in \{1, 2, 3, 4\}$. A census was conducted on Ω and the following counts where recorded.

X \ Y	1	2	3	4	
1	33	26	10	20	89
2	21	10	13	12	56
3	16	18	16	20	70
					215

(a) Record $f_{(X,Y)}$.

(b) Record $f_{Y|X}(\cdot \mid x)$ for each possible value of x.

(c) Is there a relationship between X and Y? If so what form does it take?

(2) Prove that no relationship exists between measurements X and Y defined on a population Ω iff $f_{(X,Y)}(x,y) = f_X(x) f_Y(y)$ $\forall x, y$.

(3) Suppose that $X : \Omega \to (c_0, c_k)$ and we subdivide $(c_0, c_n]$ into subintervals $(c_0, c_1], (c_1, c_2], \ldots, (c_{n-1}, c_n]$. Define the density histogram by

$$f_X^*(x) = \frac{\#\{\omega : c_{i-1} < X(\omega) \le c_i\}}{\#(\Omega)(c_i - c_{i-1})} \qquad c_{i-1} < x \le c_i$$

and $f_X(x) = 0$ when $x \in (-\infty, c_0] \cup (c_n, \infty)$.
For $c_i \le c_j$ prove that

$$\sum_{x \in (c_i, c_j]} f_X(x) = \int_{c_i}^{c_j} f_X^*(x)\,dx .$$

What is the LHS equal to?

4. Suppose that $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix}\right)$.

(a) Prove that $Y \mid X = x \sim N\left(\mu_2 + \frac{\sigma_2}{\sigma_1}\rho(x_1 - \mu_1),\ \sigma_2^2(1-\rho^2)\right)$

(b) Explain why $\sigma_2^2(1-\rho^2) < \sigma_2^2$.