Exercises on Sufficiency

Ex 1 A statistic T is sufficient for a model if $T(x_1) = T(x_2)$ implies $L(\cdot \mid x_1) = L(\cdot \mid x_2)$.

Proof: From the definition T is sufficient for M if $f_{\theta,T}(T(x))$ is a likelihood function for (M, x). But if $T(x_1) = T(x_2)$ then $f_{\theta,T}(T(x_1)) = f_\theta(T(x_2))$ which implies $L(\cdot \mid x_1) = L(\cdot \mid x_2)$

Ex 2 If $x = (x_1, \ldots, x_n) \overset{iid}{\sim}$ Bernoulli(θ) for $\theta \in [0,1]$ then show $x \mid \bar{x}$ is the uniform distribution on all sequences of 0's and 1's of length having $n\bar{x}$ 1's.

Proof: The prob. fn for $x = (x_1, \ldots, x_n)$ is

$\theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})}$ and since $n\bar{x} \sim$ binomial (n, θ)

the prob. fn for $n\bar{x}$ is $\binom{n}{n\bar{x}} \theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})}$.

Note that conditioning on $\bar{x}$ is equivalent to

conditioning on $n\bar{x}$ and so the conditional prob. fn

of $(x_1, \ldots, x_n)$ given there are k ones is

$$\frac{\theta^k(1-\theta)^{n-k}}{\binom{n}{k}\theta^k(1-\theta)^{n-k}} = \frac{1}{\binom{n}{k}} \text{ when } n\bar{x} = k$$

and is 0 otherwise. Since there are $\binom{n}{k}$ sequences of 0's and 1's containing k ones this is the uniform distribution on the set of such sequences.