IV Sufficiency

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Def A statistic $T: \mathcal{X} \to \mathcal{T}$ is sufficient for $\mathcal{M}$ if $f_{\theta T}(T(x))$ is a likelihood function for $(\mathcal{M}, x)$ $\forall x$

- recall: $f_{\theta T}(t)$ = marginal density

$$\overset{\text{discrete case}}{=} \sum_{x \in \mathcal{X}, T(x)=t} f_\theta(x)$$

Lemma 1 (Factorization) T is sufficient for $\mathcal{M}$ iff $f_\theta(x) = c(x)\, g_\theta(T(x))$ $\forall x$ for some fn $c: \mathcal{X} \to (0,\infty)$, $g_\theta: \mathcal{T} \to (0,\infty)$

Proof: $\Rightarrow$] For each x, we $f_{\theta T}(T(x)) = d(x) f_\theta(x)$ where $d(x) > 0$ so $f_\theta(x) = \frac{1}{d(x)} f_{\theta T}(T(x))$ so take $c(x) = \frac{1}{d(x)}$, $g_\theta(T(x)) = f_{\theta T}(T(x))$.

$\Leftarrow$] We have $f_{\theta T}(T(x)) = \left(\sum_{z, T(z)=T(x)} c(z)\right) g_\theta(T(x)) = \frac{c}{c(x)} f_\theta(x)$ so is a LF.

eg $x = (x_1, \dots, x_n) \overset{iid}{\sim} \text{Bernoulli}(\theta)$

$$f_\theta(x) = \prod_{i=1}^n \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})}$$

$\therefore$ by Factorization ($c(x) \equiv 1$) we have that $\bar{x}$ is a sufficient statistic.

Lemma ② T is sufficient iff $T(x_1)=T(x_2)$ implies $L(\cdot|x_1) = L(\cdot|x_2)$

Pf. Ex

Lemma ③ T is a suff stat for M iff whenever $f_{\theta T}(T(x)) > 0$ the conditional distribution $x|T(x)$ is independent of θ

Proof: (discrete case) ⇒] We have that the conditional density of x given $T(x)$ at equals

$f_\theta(x) / f_{\theta T}(T(x)) \overset{\text{Factorization}}{=} c(x) f_\theta(T(x)) / f_\theta(T(x))$

$= c(x)$.

⇐] Put $c(x) = f_\theta(x) / f_{\theta T}(T(x))$. Then

$f_{\theta T}(T(x)) = c^{-1}(x) f_\theta(x)$

and $f_{\theta T}(T(x))$ is a LF and so T is sufficient.

- note - this result has implications for model checking

- since the distribution of $x|T(x)$ is known we can compare the observed value of x with this distribution to see if it is surprising

Ex for $x = (x_1, \ldots, x_n) \overset{iid}{\sim}$ Bernoulli show $x \mid \bar{x}$ is uniform on vectors in $\{0,1\}^n$ with $n\bar{x}$ 1's.

- <u>minimal sufficiency</u>

- a sufficient statistic makes a reduction in the dimension of the data

eg - $x = (x_1, \ldots, x_n) \overset{iid}{\sim}$ Bernoulli(θ) $\theta \in [0,1]$

- $\bar{x}$ is suff so we go from n dimensions to 1 dimension.

- in making the reduction no relevant information in the data concerning the true value of θ is lost (same LF's)

- we want to make the maximum reduction possible.

Def A sufficient statistic T for M is a <u>minimal sufficient statistic</u> for M if for any other sufficient statistic T' $\exists$ function h s.t. $T(x) = h(T'(x))$

- <u>note</u> - from the model $M_T = \{f_{\theta T} : \theta \in \Theta\}$ for a sufficient statistic T and the observed value $T(x)$ we can form a likelihood fn for (M, x) via $L(\theta \mid x) = f_{\theta T}(T(x))$

- define, for $x_1, x_2 \in \mathcal{X}$ $x_1 \equiv x_2$ if
$L(\cdot|x_1) = c L(\cdot|x_2)$ for some $c > 0$

Lemma Any sufficient statistic T that satisfies $T(x_1) = T(x_2)$ iff $x_1 \equiv x_2$ is a MSS and conversely.

Proof: $\Rightarrow$] Suppose T' is a sufficient statistic. Then if $T'(x_1) = T'(x_2)$ we have $x_1 \equiv x_2$ which implies $T(x_1) = T(x_2)$. Therefore $\exists$ a fn h s.t. $T(x) = h(T'(x))$, and so T is a MSS.

$\Leftarrow$] Suppose T is a MSS and $x_1 \equiv x_2$. Now define $T'(x) = x_1$ if $x \in \{x_1, x_2\}$ and $T'(x) = x$ otherwise. Then $f_{\theta T'}(x)$ $= f_\theta(x)$ if $x \notin \{x_1, x_2\}$ and $f_{\theta T'}(x_1) = f_\theta(x_1) + f_\theta(x_2)$ $= f_\theta(x_1)(1 + 1/c)$ since $f_\theta(x_1) = c f_\theta(x_2)$. Therefore $f_{\theta T'}(x)$ is an LF and so T' is a sufficient statistic and so $\exists$ h s.t.

$T(x) = h(T'(x))$ which implies $T(x_1) = T(x_2)$

<u>**Corollary**</u> If T is a sufficient statistic and from any LF $L(\cdot|x)$ we can compute $T(x)$ then T is a MSS.

Proof: If $L(\cdot|x_1) = k L(\cdot|x_2)$ ($x_1 \equiv x_2$ then)

then $T(x_1) = T(x_2)$. If $T(x_1) = T(x_2)$

then $L(\cdot|x_1) = c L(\cdot|x_2)$ which implies

$x_1 \equiv x_2$.

<u>**eg**</u> $x = (x_1, \ldots, x_n) \sim$ Bernoulli(θ) with $\theta \in [0,1]$

- $L(\theta|x) = \theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})}$ and $T(x) = \bar{x}$ is sufficient

- $\ln L(\theta|x) = n\bar{x}\ln\theta + n(1-\bar{x})\ln(1-\theta)$

$$\frac{\partial \ln L(\theta|x)}{\partial\theta} = \frac{n\bar{x}}{\theta} - \frac{n(1-\bar{x})}{1-\theta} = 0$$

- so $(1-\theta)n\bar{x} = \theta n(1-\bar{x})$ or

$$\theta(n(1-\bar{x}) + n\bar{x}) = n\bar{x}$$

$$\theta = \bar{x}$$

$\therefore \bar{x}$ is a MSS

note $\left.\dfrac{\partial^2 \ln L(\theta|x)}{\partial\theta^2}\right|_{\theta=\bar{x}}$

$= \dfrac{-n\bar{x}}{\theta^2} - \dfrac{n(1-\bar{x})}{(1-\theta)^2} = -\dfrac{n}{\bar{x}} - \dfrac{n}{1-\bar{x}}$

$= \dfrac{-n}{\bar{x}(1-\bar{x})} < 0$

$\therefore \theta_{MLE}(x) = \bar{x}$