(d) Minimaxity and Admissibility

- recall the ingredients of a decision problem:

  model $\{f_\theta : \theta \in \Theta\}$, data $x \in \mathcal{X}$

  characteristic of interest $\psi = \Psi(\theta) \in \Psi$

  loss function $L : \Theta \times \Psi \to [0, \infty)$

  (nonrandomized) decision fn $\delta : \mathcal{X} \to \Psi$
  $\delta(x)$ is the decision

  allow for randomized decision fns s.t. $\delta(x, \cdot)$ is a prob. measure on Ψ so generate $\psi \sim \delta(x, \cdot)$ and $\delta(x, A)$ = prob. decision is in A

  risk $R(\theta, \delta) = E_\theta\left(E_{\delta(x,\cdot)}(L(\theta, \psi))\right)$

- want δ s.t. $R(\theta, \delta) \le R(\theta, \delta')$ $\forall \theta \in \Theta$ and $\forall \delta'$

- the former proof applies: except in trivial problems there is no optimal δ

- what to do?

- several strategies

(1) restrict the class of decision fns considered

eg unbiased estimators

eg size α tests

(2) choose any admissible δ: δ is admissible if $\nexists\ \delta'$ s.t. $R(\theta,\delta') \le R(\theta,\delta)$ $\forall\ \theta \in \Theta$ and $\exists\ \theta'$ s.t. $R(\theta',\delta') < R(\theta',\delta)$

eg $\delta(x) \equiv 4$ is admissible!

basically a method to rule out a δ if it is inadmissible

eg - sampling from $N_k(\underline{\mu}, \Sigma)$ where $g(\underline{\mu}, \Sigma) = \underline{\mu}$

- $L(\underline{\mu}, \underset{\sim}{t}) = \|\underline{\mu} - \underset{\sim}{t}\|^2$

- when $k \geq 3$ $\delta(\underset{\sim}{x}) = \overline{\underset{\sim}{x}}$ is inadmissible.

(3) impose a total ordering on the δ

- there are two orderings in common usage

1. <u>Minimaxity</u> - the smallest worst case behavior

δ is <u>minimax</u> if $\sup_\theta R(\theta,\delta) = \inf_{\delta'} \sup_\theta R(\theta,\delta')$

<u>Lemma</u> If δ has constant risk and is admissible, then δ is minimax

Proof: Suppose δ' is such that

$\sup_\theta R(\theta,\delta') < \sup_\theta R(\theta,\delta) = c$. But then

$R(\theta,\delta') < c \ \forall \theta$ which implies $R(\theta,\delta')$

$< R(\theta,\delta) \ \forall \theta$ which contradicts δ is

admissible.

<u>eg</u> - $x = (x_1, \ldots, x_n) \overset{iid}{\sim} N(\mu, \sigma_0^2)$, σ_0^2 known
$\Psi(\mu) = \mu$, $L(\mu, a) = \|\mu - a\|^2$

- then $\bar{x}$ is admissible and $R(\mu,\bar{x})$
$= Var_\mu(\bar{x}) = \sigma_0^2/n$ (constant risk)

$\therefore \bar{x}$ is minimax

eg - $x = (x_1, \ldots, x_n) \overset{iid}{\sim} N(\mu, \sigma^2)$ $\mu \in \mathbb{R}^1$, $\sigma^2 > 0$

$\Psi(\mu, \sigma^2) = \mu$, $L(\mu, t) = \|\mu - t\|^2$

- consider δ : $\sup_{(\mu,\sigma^2)} R((\mu,\sigma^2), \delta)$

$\geq \sup_{\sigma^2} \sup_{\mu} R((\mu,\sigma^2), \delta)$: b

by previous example

$\geq \sup_{\sigma^2} \sup_{\mu} R((\mu,\sigma^2), \bar{x}) = \sup_{\sigma^2} \frac{\sigma^2}{n} = \infty$

$\therefore$ every δ is minimax!

2. Bayes rules

- we will discuss Bayesian inference now.