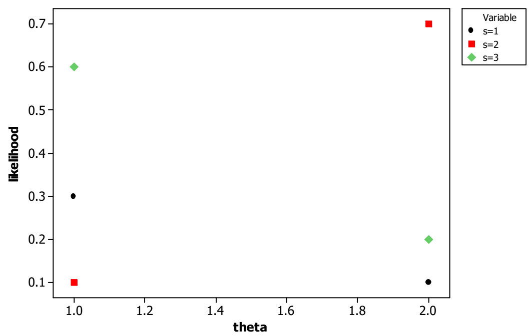
**6.1.8**
(a) The three likelihood functions are as follows.




(b) Since $L(1|1)/L(2|1) = 0.3/0.1 = 3 = L(1|3)/L(2|3)$ and $L(1|2)/L(2|2) = 0.1/0.7 = 1/7 \neq 3$, a statistic $T : S \to \{1, 2\}$ given by $T(1) = T(3) = 1$ and $T(2) = 2$ is a sufficient statistic.

**6.1.19** The likelihood function is given by

$$\begin{aligned} L(\theta \,|\, x_1, ..., x_n) &= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_0)} (\theta x_i)^{\alpha_0 - 1} \exp\{-\theta x_i\}\, \theta \\ &= \Gamma^{-n}(\alpha_0) \left(\prod x_i\right)^{\alpha - 1} \theta^{n\alpha_0} \exp(-\theta n \bar{x}). \end{aligned}$$

By the factorization theorem $\bar{x}$ is a sufficient statistic. The logarithm of the likelihood is given by $\ln L(\theta \,|\, x_1, ..., x_n) = \ln\{\Gamma^{-n}(\alpha_0) \left(\prod x_i\right)^{\alpha_0 - 1}\} + n\alpha \ln\theta - \theta n \bar{x}$. Differentiating this and setting it equal to 0, we obtain $\theta = \alpha/\bar{x}$. So given a likelihood function, we can determine $\bar{x}$ and this proves that $\bar{x}$ is minimal sufficient.

**6.1.20** The likelihood function is given by $L(\theta \,|\, x_1, ..., x_n) = \theta^{-n} I_{[x_{(n)}, \infty)}(\theta)$ when $\theta > 0$. By the factorization theorem $x_{(n)}$ is a sufficient statistic. Now notice that the likelihood function is 0 to the left of $x_{(n)}$ and positive to the right. So given the likelihood, we can determine $x_{(n)}$ and it is minimal sufficient.

**6.2.20** First, recall that the MLE for $\mu$ is $\bar{x}$ (Example 6.2.2). The parameter of interest now is $\psi(\mu) = P_\mu(X < 1) = \Phi(1 - \mu)$, where $\Phi$ is the cdf of a $N(0, 1)$. Since $\Phi(1 - \mu)$ is a strictly decreasing function $\mu$, then $\psi$ is a 1-1 function of $\mu$. Hence, we can apply Theorem 6.2.1 and conclude that $\hat{\psi} = \Phi(1 - \bar{x})$ is the MLE.

**6.3.25** Using $c(x_1, ...., x_n) = \bar{x} + k(\sigma_0/\sqrt{n})$, we have that $k$ satisfies

$$P\left(\mu \leq \bar{x} + k\left(\sigma_0/\sqrt{n}\right)\right) = P\left(\frac{\bar{x} - \mu}{\sigma_0/\sqrt{n}} \geq -k\right) = P(Z \geq -k) \geq \gamma$$

So $k = -z_{1-\gamma} = z_\gamma$, i.e., the $\gamma$-percentile of a $N(0, 1)$ distribution.

**6.3.26** The P-value for testing $H_0 : \mu \le \mu_0$ is given by

$$\begin{aligned}\max_{\mu\in H_0} P_\mu\left(\frac{\bar{X}-\mu}{\sigma_0/\sqrt{n}} > \frac{\bar{x}_o-\mu}{\sigma_0/\sqrt{n}}\right) &= \max_{\mu\in H_0} P\left(Z > \frac{\bar{x}_o-\mu}{\sigma_0/\sqrt{n}}\right) \\ &= \max_{\mu\in H_0}\left(1-\Phi\left(\frac{\bar{x}_o-\mu}{\sigma_0/\sqrt{n}}\right)\right)\end{aligned}$$

Since $\left(1-\Phi\left(\left(\bar{x}_o-\mu\right)/\left(\sigma_0/\sqrt{n}\right)\right)\right)$ is an increasing function of $\mu$, its maximum is at $\mu = \mu_0$.

**6.3.27** The form of the power function associated with the above hypothesis assessment procedure is given by

$$\begin{aligned}\beta(\mu) &= P_\mu\left(1-\Phi\left(\frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}}\right) < \alpha\right) = P_\mu\left(\Phi\left(\frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}}\right) > 1-\alpha\right) \\ &= P_\mu\left(\frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}} > z_{1-\alpha}\right) = P_\mu\left(\frac{\bar{X}-\mu}{\sigma_0/\sqrt{n}} > \frac{\mu_0-\mu}{\sigma_0/\sqrt{n}} + z_{1-\alpha}\right) \\ &= 1-\Phi\left(\frac{\mu_0-\mu}{\sigma_0/\sqrt{n}} + z_{1-\alpha}\right).\end{aligned}$$