7.1.16 Suppose that $X_\tau \sim N\left(\mu_0, \tau^2\right)$. Then $P(X_\tau < x) = \Phi\left(\left(x - \mu_0\right)/\tau\right) \to \Phi\left(0\right) = 1/2$ for every x and this is not a distribution function.

7.1.17 First, observe that the posterior density of θ given $x_1, ...x_n$ is $\pi\left(\theta \mid x_1, ...x_n\right) \propto \pi\left(\theta\right) \prod_{i=1}^{n} f_\theta\left(x_i\right)$. Using this as the prior density to obtain the posterior density of θ given $x_{n+1}, ...x_{n+m}$, we get $\pi\left(\theta, x_1, ...x_n \mid x_{n+1}, ...x_{n+m}\right) \propto \pi\left(\theta\right) \prod_{i=1}^{n} f_\theta\left(x_i\right) \prod_{i=n+1}^{m+n} f_\theta\left(x_i\right)$, and this is the same as the posterior density of θ given $x_1, ...x_n, x_{n+1}, ...x_{n+m}$.

7.2.17 From the equation $BF(A) = [\Pi(A|s)/(1-\Pi(A|s))]/[\Pi(A)/(1-\Pi(A))]$, we get $\Pi(A|s) = 1/[1 + BF(A)/[\Pi(A)/(1-\Pi(A))]]$. Both statisticians' Bayes factor equals $BF(A) = 100$. The prior odds of Statistician I is $\Pi(H_0)/(1-\Pi(H_0)) = (1/2)/(1/2) = 1$. Thus Statistician I's posterior probability is $\Pi(H_0|s) = 1/[1+(1)100] = 1/101 = 0.0099$. The prior odds of Statistician II is $\Pi(H_0)/(1-\Pi(H_0)) = (1/4)/(3/4) = 1/3$ and the posterior probability is $\Pi(H_0|s) = 1/[1+(1/3)100] = 3/103 = 0.0292$. Hence, Statistician II has the bigger posterior belief in H_0.

7.1.18 The joint density of $(\theta, x_1, ...x_n)$ is given by

$$\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} \theta^{n\bar{x}+\alpha-1}\left(1-\theta\right)^{n(1-\bar{x})+\beta-1}$$

and integrating out θ gives the marginal probability function for $(x_1, ...x_n)$ as $m\left(x_1, ...x_n\right) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(n\bar{x}+\alpha)\Gamma(n(1-\bar{x})+\beta)}{\Gamma(\alpha+\beta+n)}$ for $(x_1, ...x_n) \in \{0,1\}^n$.

To generate from this distribution we can first generate $\theta \sim \text{Beta}(\alpha, \beta)$ and then generate $x_1, ...x_n$ i.i.d. from the Bernoulli(θ) distribution.

7.2.18 Note that a credible set is an acceptance region and the compliment of γ-credible set is a $(1-\gamma)$ rejection region. Since $\psi(\theta) = 0 \in (-3.3, 2.6)$, the P-value must be greater than $1 - 0.95 = 0.05$.

7.2.21 The likelihood function is given by $L\left(\theta \,|\, x_1, .., x_n\right) = \theta^{-n} I_{\left(x_{(n)}, \infty\right)}\left(\theta\right)$ and the prior is $I_{(0,1)}\left(\theta\right)$, so the posterior is

$$\frac{\theta^{-n} I_{\left(x_{(n)},1\right)}\left(\theta\right)}{\int_{x_{(n)}}^{1} \theta^{-n}\, d\theta} = \frac{\theta^{-n} I_{\left(x_{(n)},1\right)}\left(\theta\right)}{(n-1)\left(x_{(n)}^{1-n} - 1\right)}.$$

Since this density strictly increases in $\left(x_{(n)}, 1\right)$ and HPD interval is of the form $(c, 1)$, c is determined by

$$\gamma = \int_{c}^{1} \frac{\theta^{-n} I_{\left(x_{(n)},1\right)}\left(\theta\right)}{(n-1)\left(x_{(n)}^{1-n} - 1\right)}\, d\theta = \frac{c^{1-n} - 1}{x_{(n)}^{1-n} - 1},$$

so $c = \left\{1 + \gamma\left(x_{(n)}^{1-n} - 1\right)\right\}^{1/(1-n)}$.

7.2.23 Let $\psi\left(\mu,\sigma^2\right) = \mu + \sigma z_{0.75} = \mu + \left(1/\sigma^2\right)^{-1/2} z_{0.75}$ and $\lambda = \lambda\left(\mu,\sigma^2\right) = 1/\sigma^2$, so

$$J\left(\theta\left(\psi,\lambda\right)\right) = \left|\det\begin{pmatrix} \frac{\partial\psi}{\partial\mu} & \frac{\partial\psi}{\partial\left(\frac{1}{\sigma^2}\right)} \\ \frac{\partial\lambda}{\partial\mu} & \frac{\partial\lambda}{\partial\left(\frac{1}{\sigma^2}\right)} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & -\frac{1}{2}z_{0.75}\left(\frac{1}{\sigma^2}\right)^{-\frac{3}{2}} \\ 0 & 1 \end{pmatrix}\right| = 1.$$

Therefore, the posterior density of ψ is given by

$$\int_0^\infty \frac{1}{\sqrt{2\pi}}\left(n+\frac{1}{\tau_0^2}\right)^{1/2}\lambda^{1/2}\exp\left(-\frac{\lambda}{2}\left(n+\frac{1}{\tau_0^2}\right)\left(\left(\psi_0-\lambda^{-1/2}z_{0.75}\right)-\mu_x\right)^2\right)$$
$$\times\frac{\left(\beta_x\right)^{\alpha_0+n/2}}{\Gamma\left(\alpha_0+n/2\right)}\lambda^{\alpha_0+n/2-1}\exp\left(-\beta_x\lambda\right)\,d\lambda.$$

which is a difficult integral to evaluate.

7.2.28 Since the variance of a $t(\lambda)$ distribution is $\lambda/\left(\lambda-2\right)$, the posterior variance of μ is given by

$$Var\left(\mu_x+\sqrt{\frac{1}{n+2\alpha_0}}\sqrt{\frac{2\beta_x}{n+1/\tau_0^2}}t(n+2\alpha_0)\right)$$
$$=\left(\sqrt{\frac{1}{n+2\alpha_0}}\sqrt{\frac{2\beta_x}{n+1/\tau_0^2}}\right)^2\frac{n+2\alpha_0}{n+2\alpha_0-2}=\left(\frac{2\beta_x}{n+1/\tau_0^2}\right)\left(\frac{1}{n+2\alpha_0-2}\right).$$

7.2.32 We can write $X_{n+1} = \mu + \sigma U$, where $U \sim N(0,1)$ independent of $X_1,\ldots,X_n,\mu,\sigma$. We also have that $\mu = \mu_x + \left(n+1/\tau_0^2\right)^{-1/2}\sigma Z$, where $Z \sim N(0,1)$ is independent of $X_1,\ldots,X_n,\sigma$. Therefore, we can write

$$X_{n+1} = \mu_x + \left(n+1/\tau_0^2\right)^{-1/2}\sigma Z + \sigma U$$
$$= \mu_x + \sigma\left\{\left(n+1/\tau_0^2\right)^{-1/2}Z+U\right\} = \mu_x + \left\{\left(n+1/\tau_0^2\right)^{-1}+1\right\}^{1/2}\sigma W$$

where

$$W = \left\{\left(n+1/\tau_0^2\right)^{-1}+1\right\}^{-1/2}\left\{\left(n+1/\tau_0^2\right)^{-1/2}Z+U\right\}$$
$$= \frac{X_{n+1}-\mu_x}{\left\{\left(n+1/\tau_0^2\right)^{-1}+1\right\}^{1/2}\sigma} \sim N(0,1)$$

is independent of $X_1,\ldots,X_n,\sigma$. Therefore, just as in Example 7.2.1,

$$T = \frac{W}{\sqrt{\left(2\frac{\beta_x}{\sigma^2}\right)/\left(2\alpha_0+n\right)}}$$
$$= \frac{X_{n+1}-\mu_x}{\left\{\left(n+1/\tau_0^2\right)^{-1}+1\right\}^{1/2}\left(\left(2\beta_x\right)/\left(2\alpha_0+n\right)\right)^{1/2}} \sim t\left(2\alpha_0+n\right).$$

7.2.34 The prior predictive probability measure for the data s with a mixture of Π_1 and Π_2 prior distributions is given by

$$\begin{aligned}
&m(s) = E_\Pi\left(f_\theta(s)\right) = \sum_\theta f_\theta(s)\,\Pi(\{\theta\}) \\
&= \sum_\theta f_\theta(s)\left(p\Pi_1(\{\theta\}) + (1-p)\,\Pi_2(\{\theta\})\right) \\
&= p\sum_\theta f_\theta(s)\,\Pi_1(\{\theta\})\,(1-p)\sum_\theta f_\theta(s)\,\Pi_2(\{\theta\}) \\
&= pf_{\theta_0}(s) + (1-p)\sum_\theta f_\theta(s)\,\Pi_2(\{\theta\}) = pm_1(s) + (1-p)\,m_2(s)\,.
\end{aligned}$$

The posterior probability measure is given by

$$\begin{aligned}
&\Pi(A\,|\,s) = \sum_{\theta\in A}\frac{f_\theta(s)\,\Pi(\{\theta\})}{m(s)} = \sum_{\theta\in A}\frac{f_\theta(s)\left(p\Pi_1(\{\theta\}) + (1-p)\,\Pi_2(\{\theta\})\right)}{pm_1(s) + (1-p)\,m_2(s)} \\
&= \frac{pm_1(s)}{pm_1(s) + (1-p)\,m_2(s)}\sum_{\theta\in A}\frac{f_\theta(s)\,\Pi_1(\{\theta\})}{m_1(s)} \\
&\quad + \frac{(1-p)\,m_2(s)}{pm_1(s) + (1-p)\,m_2(s)}\sum_{\theta\in A}\frac{f_\theta(s)\,\Pi_2(\{\theta\})}{m_2(s)} \\
&= \frac{pm_1(s)}{pm_1(s) + (1-p)\,m_2(s)}\Pi_1(A\,|\,s) + \frac{(1-p)\,m_2(s)}{pm_1(s) + (1-p)\,m_2(s)}\Pi_2(A\,|\,s)\,.
\end{aligned}$$

7.3.11
(a) We have that $\frac{1}{n}\ln\left(L\left(\hat\theta\,|\,x_1,\ldots,x_n\right)\pi\left(\hat\theta\right)\right) = \frac{1}{n}\sum_{i=1}^n \ln L\left(\hat\theta\,|\,x_i\right) + \frac{1}{n}\ln\pi\left(\hat\theta\right) \overset{a.s}{\to} E_\theta\left(\ln L(\theta\,|\,X)\right) = I(\theta)$ by the strong law of large numbers.
(b) Then from the results of part (a) we have that, denoting the true value of θ by θ_0,

$$\frac{\theta - \hat\theta\left(X_1,\ldots,X_n\right)}{\hat\sigma\left(X_1,\ldots,X_n\right)/\sqrt{n}} \overset{a.s}{\to} \sqrt{nI(\theta_0)}\,(\theta-\theta_0)$$

when $\theta \sim \Pi\left(\cdot\,|\,X_1,\ldots,X_n\right)$. This implies that when the sample size is large then inferences will be independent of the prior.

7.3.15 Suppose that the posterior expectation of ψ exists. Then by the theorem of total expectation we have that

$$
\begin{aligned}
&E(\psi \mid x_1, \ldots, x_n) = E\left(\frac{\sigma}{\mu} \mid x_1, \ldots, x_n\right) \\
&= E\left(\frac{\sigma}{\mu}\left(I_{(-\infty,0)}(\mu) + I_{(0,\infty)}(\mu)\right) \mid x_1, \ldots, x_n\right) \\
&= E\left(\frac{\sigma}{\mu} I_{(-\infty,0)}(\mu) \mid x_1, \ldots, x_n\right) + E\left(\frac{\sigma}{\mu} I_{(0,\infty)}(\mu) \mid x_1, \ldots, x_n\right) \\
&= E\left(E\left(\frac{\sigma}{\mu} I_{(-\infty,0)}(\mu) \mid \sigma, x_1, \ldots, x_n\right) \mid x_1, \ldots, x_n\right) \\
&\quad + E\left(E\left(\frac{\sigma}{\mu} I_{(0,\infty)}(\mu) \mid \sigma, x_1, \ldots, x_n\right) \mid x_1, \ldots, x_n\right)
\end{aligned}
$$

and reasoning as in Problem 7.2.24, we have that $E\left(\frac{\sigma}{\mu} I_{(-\infty,0)}(\mu) \mid \sigma, x_1, \ldots, x_n\right)$ $= -\infty$ and $E\left(\frac{\sigma}{\mu} I_{(-\infty,0)}(\mu) \mid \sigma, x_1, \ldots, x_n\right) = \infty$, so $E(\psi \mid x_1, \ldots, x_n) = \infty - \infty$ which is undefined.

7.4.3
(a) First, we compute the prior predictive for the data as follows.

$$
m_\tau(1,1,3) = \sum_{\theta=1}^{2} \pi(\theta) f_\theta(1,1,3) = \begin{cases} \frac{1}{2}\left(\frac{1}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{2}\right)^2 \frac{1}{8} = \frac{59}{1728} & \tau = 1 \\ \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{2}{3}\left(\frac{1}{2}\right)^2 \frac{1}{8} = \frac{43}{1296} & \tau = 2 \end{cases}
$$

The maximum value of the prior predictive is obtained when $\tau = 1$, therefore we choose the first prior.
(b) The posterior of θ given $\tau = 1$ is

$$
\pi_1(\theta \mid 1,1,3) = \begin{cases} \frac{\frac{1}{2}\left(\frac{1}{3}\right)^3}{\frac{59}{1728}} = \frac{32}{59} & \theta = a \\ \frac{\frac{1}{2}\left(\frac{1}{2}\right)^2 \frac{1}{8}}{\frac{59}{1728}} = \frac{27}{59} & \theta = b. \end{cases}
$$

7.4.16 From Exercise 6.5.1 the Fisher information is $n/2\sigma^4$. Therefore, Jeffreys' prior is given by $1/\sigma^2$.

7.4.17 We use the prior $1/\sigma^2$. The posterior distribution is proportional to

$$
\begin{aligned}
&\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left(-\frac{n}{2\sigma^2}(\bar{x} - \mu)^2\right) \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \frac{1}{\sigma^2} \\
&= \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \exp\left(-\frac{n}{2\sigma^2}(\bar{x} - \mu)^2\right) \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}}.
\end{aligned}
$$

So the posterior distribution of (μ, σ^2) is given by $\mu \mid \sigma^2, x_1, ..., x_n \sim N(\bar{x}, \sigma^2/n)$ and $1/\sigma^2 \mid x_1, ...x_n \sim$Gamma$\left(\frac{n+3}{2}, \frac{n-1}{2}s^2\right)$.