STAC58S: 2015 Final Exam

Name: Solutions

Student Number:

Time: 3 hours
Instructions: Write your answers on the exam paper. You do not have to fully evaluate numerical results for questions that ask for these as full marks will be given when it is clear that you have the correct expression.
Aids: The exam is open book. Students may use any notes, books and calculators in writing this exam. You are free to refer to any results derived in class or on the assignments.

1. A population Ω consists of 10 objects and two measurements $(X(\omega), Y(\omega))$ are associated with each $\omega \in \Omega$. Suppose that the following measurements were obtained.

ω	1	2	3	4	5	6	7	8	9	10
$X(\omega), Y(\omega)$	$0, 2$	$1, 1$	$0, 3$	$0, 1$	$0, 1$	$0, 2$	$1, 2$	$1, 2$	$1, 3$	$0, 1$

(a) (5 marks) Determine the joint distribution of (X, Y).

(x, y)	$f_{(X,Y)}(x, y)$
$0, 1$	$3/10$
$0, 2$	$2/10$
$0, 3$	$1/10$
$1, 1$	$1/10$
$1, 2$	$2/10$
$1, 3$	$1/10$

(b) (5 marks) Indicate why there is, or is not, a relationship between X and Y.

y	$f_Y(y \mid X=0)$	$f_Y(y \mid X=1)$
1	3/6	1/4
2	2/6	2/4
3	1/6	1/4

Since these distributions are different there is a relationship between X and Y.

(c) (5 marks) Suppose for some reason you were not able to observe the measurements on each of the $\omega \in \Omega$ but you are allowed to take the measurements of at most 5 members of Ω. Discuss how you would choose the elements of Ω and justify your answer.

We would use a random system that generated a value $i \in \{1, 2, \ldots, 10\}$ uniformly. Then 5 values would be generated $i_1, \ldots, i_5$ from this distribution and the measurements $(X(\omega_{i_j}), Y(\omega_{i_j}))$ $j = 1, \ldots, 5$ taken. This guarantees the objectivity of the observed data. Since it is a small population it is better to do the sampling without replacement.

3. Suppose we have a sample $x = (0,1,0,1,0)$ from a Bernoulli(θ) distribution where $\theta \in [0,1]$ is unknown.

(a) (5 marks) Determine the likelihood function and explain why likelihood inferences only depend on likelihood ratios.

$L(\theta \mid \bar{x}) = \theta^2 (1-\theta)^3$ for $\theta \in [0,1]$.

The likelihood function is proportional to the probabilities of the observed data as θ varies. It is the relative sizes of these probabilities that determine the degree to which one value of θ is supported over another.

(b) (5 marks) Determine the MLE of θ and the likelihood interval $C_{1/2}(x)$. How is $C_{1/2}(x)$ used in an application?

Through differentiating $\log L(\theta \mid x)$ and setting equal to 0 obtain $\theta_{MLE}(x) = \arg\sup_{\theta} L(\theta \mid x) = 2/5$

since $L(\cdot \mid \bar{x})$ peaks at $2/5$, and

$L(\theta_{MLE}(x) \mid x) = \frac{2^2 \cdot 3^3}{5^5}$ Then

$C_{1/2}(x) = \{\theta : \theta^2 (1-\theta)^3 \geq \frac{1}{2} \frac{2^2 3^3}{5^5} = \frac{54}{3125}\}$

$= \{\theta : \theta^5 - \theta^2 + \frac{514}{3125} \leq 0\}$

$= [\theta_1, \theta_2]$ where θ_1, θ_2 are the roots of $\theta^5 - \theta^2 + \frac{514}{3125}$ in $[0,1]$.

We look at the length of $C_{1/2}(x)$ to assess the accuracy of $\theta_{MLE}(x)$.

(c) (5 marks) Suppose that

$$\psi = \Psi(\theta) = \begin{cases} 1 & \theta \in [0, 1/4] \\ 2 & \theta \in (1/4, 3/4) \\ 3 & \theta \in [3/4, 1]. \end{cases}$$

Determine the profile likelihood for ψ, the profile MLE and $C_{\Psi,1/2}(x)$.

$L^{\Psi}(1 \mid x) = \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^3$, $L^{\Psi}(2 \mid x) = \left(\frac{2}{5}\right)^2 \left(\frac{3}{5}\right)^3$

$L^{\Psi}(3 \mid x) = \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^3$ Clearly $\psi_{MLE}(x) = 2$

with $L^{\Psi}(\psi_{MLE}(x) \mid x) = \frac{1}{2}\left(\frac{2}{5}\right)^2 \left(\frac{3}{5}\right)^3$. Then

$C_{1/2}(x) = \{\psi \mid L^{\Psi}(\psi \mid x) \geq \frac{1}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^3\}$

$= \{1, 2\}$

4. (a) (10 marks) Suppose that $x = (x_1, \ldots, x_n)$ is a sample from the Poisson(λ) where $\lambda > 0$ is unknown. Prove that the statistic $T(x) = \sum_{i=1}^n x_i$ is a minimal sufficient statistic for this model.

We have $f_\lambda(x) = \dfrac{\lambda^{T(x)} e^{-n\lambda}}{x_1! \, x_2! \cdots x_n!} = \left(\lambda^{T(x)} e^{-n\lambda}\right)\left(\dfrac{1}{x_1! \, x_2! \cdots x_n!}\right)$

and so by factorization $T(x)$ is sufficient.

Now $\log f_\lambda(x) = T(x) \log \lambda - n\lambda + \log c$, so

$\dfrac{\partial \log f_\lambda(x)}{\partial \lambda} = \dfrac{T(x)}{\lambda} - n$ and setting this equal

to 0 and solving for λ gives $\lambda = T(x)/n$.

Therefore, we can calculate $T(x)$ from the LF and T is a MSS.

(b) (5 marks) Determine the distribution of T in (a).

The mgf is $m_T(t) = E\left(e^{t\sum_{i=1}^{n} x_i}\right)$

independence $= \prod_{i=1}^{n} E(e^{tx_i})$ and when $X \sim \text{Poisson}(\lambda)$

$E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} e^{-\lambda} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$

$= \exp\{-\lambda + \lambda e^t\}$. Therefore $m_T(t)$

$= \exp\{-n\lambda + n\lambda e^t\}$ and the Uniqueness

Thm implies that $T \sim \text{Poisson}(n\lambda)$.

5. Suppose that $x = (x_1, \ldots, x_n)$ is a sample from the Uniform$(0, \theta)$ distribution, where $\theta > 0$ is unknown, and interest is in estimating θ.

(a) (5 marks) Determine an unbiased estimator of θ based on $\bar{x}$.

When $X \sim$ Uniform$(0,\theta)$ then $E(X) = \theta/2$. Therefore $E(\bar{X}) = \theta/2$ and so $2\bar{X}$ is an unbiased estimator of θ.

(b) (5 marks) Prove that the largest order statistic $x_{(n)}$ is a minimal sufficient statistic for this problem.

We have $L(\theta \mid x) = \begin{cases} 0 & \theta < x_{(n)} \\ \theta^{-n} & \theta \geq x_{(n)} \end{cases}$

$= I_{[x_{(n)}, \infty)}(\theta)/\theta^n$ and so by factorization $x_{(n)}$ is sufficient. Also $\hat{\theta}_{MLE}(x) = x_{(n)}$ and so $x_{(n)}$ is a MSS.

(c) (5 marks) Prove that $x_{(n)}$ is complete for this problem. (Hint: use the fact that if the integral of an integrand equals 0 over every interval, then the integrand is 0.)

The cdf of $x_{(n)}$ is $F_\theta(x_{(n)}) = P_\theta(X_1 \le x_{(n)}, \ldots, X_n \le x_{(n)})$
$= \prod_{i=1}^{n} P_\theta(X_i \le x_{(n)}) = x_{(n)}^n / \theta^n$ for $0 \le x_{(n)} \le \theta$ and so
$x_{(n)}$ has pdf $f_\theta(x_{(n)}) = n x_{(n)}^{n-1} / \theta^n$. Now $\int_0^\theta h(x) n x^{n-1}/\theta^n \, dx$
$= 0$ for every $\theta > 0$ iff $\int_0^\theta h(x) x^{n-1} dx = 0 \quad \forall \theta > 0$.
Therefore, for any $0 < a < b$, $\int_a^b h(x) x^{n-1} dx = \int_0^b h(x) x^{n-1} dx$
$- \int_0^a h(x) x^{n-1} dx = 0$. Then by the hint we must
have $h(x) = 0 \quad \forall x$.

(d) (5 marks) Determine the optimal unbiased estimator of θ.

$$E_\theta(x_{(n)}) = \int_0^\theta \frac{n x^n}{\theta^n} = \frac{n}{n+1} \frac{x^{n+1}}{\theta^n} \Big|_0^\theta = \frac{n}{n+1}\theta.$$

Therefore, $E_\theta\left(\frac{n+1}{n} x_{(n)}\right) = \theta$ and $\frac{n+1}{n} x_{(n)}$ is unbiased for θ. By completeness $\frac{n+1}{n} x_{(n)}$ is the only function of $x_{(n)}$ that is unbiased for θ and therefore it must be optimal.

(e) (5 marks) What is the conditional expectation of $\bar{x}$ given $x_{(n)}$. Justify your answer.

$E_\theta(E(2\bar{x}|x_{(n)})) = E_\theta(2\bar{x}) = \theta$ so $E(2\bar{x}|x_{(n)})$ is unbiased for θ and is a function of $x_{(n)}$ it must be equal to $\frac{n+1}{n} x_{(n)}$ by completeness of $x_{(n)}$. Therefore $E(\bar{x}|x_{(n)})$

$= \frac{2(n+1)}{n} x_{(n)}$

6. Suppose that $x = (x_1, \ldots, x_n)$ is a sample from the $N(0, \sigma^2)$ distribution where σ^2 is unknown.

(a) (10 marks) Determine the UMP size α test for the hypothesis testing problem $H_0 : \sigma^2 \le \sigma_0^2$ versus $H_a : \sigma^2 > \sigma_0^2$.

We have $f_{\sigma^2}(x) = (2\pi)^{-\frac{n}{2}} \left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right\}$

and so by Factorization $\sum_{i=1}^n x_i^2$ is sufficient. The MLE of σ^2 is $\frac{1}{n}\sum_{i=1}^n x_i^2$ and $T(x) = \sum_{i=1}^n x_i^2$ is MSS.

Now $\frac{1}{\sigma^2}\sum_{i=1}^n x_i^2 \sim$ chi-squared(n). By the FL the MP size α test of σ_0^2 vs σ_1^2 is of the form

$$\varphi(T(x)) = \begin{cases} 1 & f_{\sigma_1^2, T}(T(x))/f_{\sigma_0^2, T}(T(x)) > k \\ \gamma & = k \\ 0 & < k \end{cases}$$

and

$$f_{\sigma_1^2, T}(T(x)) / f_{\sigma_0^2, T}(T(x)) = \frac{(\Gamma(\frac{n}{2}))^{-1} 2^{-\frac{n}{2}} (T(x))^{\frac{n}{2}-1} (\sigma_1^2)^{-\frac{n}{2}} e^{-T(x)/2\sigma_1^2}}{(\Gamma(\frac{n}{2}))^{-1} 2^{-\frac{n}{2}} (T(x))^{\frac{n}{2}-1} (\sigma_0^2)^{-\frac{n}{2}} e^{-T(x)/2\sigma_0^2}}$$

$= (\sigma_0^2/\sigma_1^2)^{n/2} \exp\left\{-\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) T(x)\right\}$ and since $\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) < 0$ this implies

$$\varphi(T(x)) = \begin{cases} 1 & T(x)/\sigma_0^2 > \chi^2_{1-\alpha}(n) \\ 0 & \text{otherwise.} \end{cases}$$

and $\gamma = 0$ since T has a continuous distribution. Since the test doesn't involve σ_1^2 it is UMP size α for $H_0 : \sigma^2 = \sigma_0^2$ vs $H_a : \sigma^2 > \sigma_0^2$. Since

$P_{\sigma^2}(T(x)/\sigma_0^2 > \chi^2_{1-\alpha}(n)) = P_{\sigma^2}(T(x)/\sigma^2 > \frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}(n)) \le \alpha$ when $\sigma^2 \le \sigma_0^2$ it is [illegible]

(b) (5 marks) Suppose that a prior is placed on σ^2 given by $1/\sigma^2 \sim \text{gamma}_{rate}(\alpha_0, \beta_0)$. Determine the posterior distribution of $1/\sigma^2$.

The joint density of $(T(x), 1/\sigma^2)$ is proportional to.
$\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} (T(x))^{n/2-1} e^{-T(x)/2\sigma^2} \left(\frac{1}{\sigma^2}\right)^{\alpha_0 - 1} e^{-\beta_0/\sigma^2}$ and as a function of σ^2
$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+\alpha_0-1} \exp\{-(\frac{T(x)}{2}+\beta_0)/\sigma^2\}$ and so the conditional (posterior) of $1/\sigma^2$ is $\text{gamma}(\frac{n}{2}+\alpha_0, (\frac{T(x)}{2}+\beta_0))$

(b) (5 marks) Determine the relative belief ratio for σ^2 and explain why this can be determined from the relative belief ratio of $1/\sigma^2$.

$$RB\left(\tfrac{1}{\sigma^2} \,\middle|\, x\right) = \frac{\pi(1/\sigma^2 \mid x)}{\pi(1/\sigma^2)}$$

$$= \frac{\frac{1}{\Gamma(\frac{n}{2}+\alpha_0)} \left(\frac{T(x)}{2}+\beta_0\right)^{\frac{n}{2}+\alpha_0} \exp\{-(\frac{T(x)}{2}+\beta_0)\frac{1}{\sigma^2}\}}{\frac{1}{\Gamma(\alpha_0)} \beta_0^{\alpha_0} \exp\{-\beta_0(\frac{1}{\sigma^2})\}}$$

$$= \frac{\Gamma(\alpha_0)}{\Gamma(\frac{n}{2}+\alpha_0)} \beta_0^{-\alpha_0} \left(\frac{T(x)}{2}+\beta_0\right)^{\frac{n}{2}+\alpha_0} \exp\{-\frac{T(x)}{2}(\frac{1}{\sigma^2})\}.$$

Also $RB(\sigma^2 \mid x) = RB(\frac{1}{\sigma^2} \mid x)$ since relative belief ratios are invariant.