STAC58S: 2017 Final Exam

Name:

Student Number:

Time: 3 hours
Instructions: Write your answers on the exam paper. You do not have to fully evaluate numerical results for questions that ask for these as full marks will be given when it is clear that you have the correct expression.
Aids: The exam is open book. Students may use any notes, books and calculators in writing this exam. You are free to refer to any results derived in class or on the assignments.

1. Measurements $X : \Omega \to \{1, 2, 3\}$ and $Y : \Omega \to \{1, 2\}$ are defined on a population Ω. Suppose a census is performed and the following table of counts is obtained.

	$Y = 1$	$Y = 2$
$X = 1$	100	300
$X = 2$	200	100
$X = 3$	100	200

(a) (5 marks) Record the joint distribution of (X, Y).

$\#(\Omega) = 1000$

(x,y)	$f_{(X,Y)}(x,y)$
$(1,1)$	$100/1000 = 0.1$
$(2,1)$	$200/1000 = 0.2$
$(3,1)$	$100/1000 = 0.1$
$(1,2)$	$300/1000 = 0.3$
$(2,2)$	$100/1000 = 0.1$
$(3,2)$	$200/1000 = 0.2$

(d) (5 marks) For a randomly selected $\omega \in \Omega$ determine the distribution of $W = X + Y$.

$f_W(w) = P(W=w) = P(\{(x,y) : x+y = w\})$

w	$f_W(w)$
2	.1
3	.5
4	.2
5	.2

(e) (5 marks) What is the conditional distribution of W given that $X = 1$? .

w	$f_{W\mid X}(w\mid 1)$
2	1/4
3	3/4
4	0
5	0

2. Suppose that a statistical model for a response $x \in \{1, 2, 3\}$ is given by the following table.

x	$\theta = a$	$\theta = b$	$\theta = c$
1	1/2	1/3	1/5
2	1/2	2/3	4/5

(a) (5 marks) Suppose that a sample of size $n = 2$ is taken. Record the model for the sample.

(x_1, x_2)	$\theta = a$	$\theta = b$	$\theta = c$
$(1,1)$	1/4	1/9	1/25
$(1,2)$	1/4	2/9	4/25
$(2,1)$	1/4	2/9	4/25
$(2,2)$	1/4	4/9	16/25

(b) (5 marks) If the sample $(x_1, x_2) = (1, 2)$ is observed, what is the MLE of θ and the 0.9-likelihood region?

The likelihood is

θ	a	b	c
$L(\theta \mid (1,2))$	1/4	2/9	4/25

$\therefore$ MLE is $\theta_{MLE}((1,2)) = a$ with

$L(\theta_{MLE}(1,2) \mid (1,2)) = .1/4$

$$C_{.9}((1,2)) = \left\{ \theta : \frac{L(\theta \mid (1,2))}{1/4} \geq .1 \right\}$$

$$= \{ \theta : L(\theta \mid (1,2)) \geq .025 \} = \{a, b, c\}.$$

(c) (5 marks) Now suppose interest is in making inference about

$$\psi = \Psi(\theta) = \begin{cases} 0 & \theta \in \{a, b\}, \\ 1 & \theta \in \{c\}. \end{cases}$$

Determine the profile likelihood based on the sample as well as the profile likelihood MLE of ψ.

$$L^{\Psi}(0 \mid (1,2)) = \sup_{\{a,b\}} \{L(a \mid (1,2)), L(b \mid (1,2))\} = 1/4$$

$$L^{\Psi}(1 \mid (1,2)) = L(c \mid (1,2)) = 4/25$$

$$\therefore \psi_{MLE}(1,2) = 0$$

(d) (5 marks) In this example is the profile likelihood a likelihood. Explain why or why not.

There is no function T defined on the sample space such that $L^{\Psi}(0 \mid (1,2)) = g P_{\theta,T}(A)$ when $\theta \in \{a,b\}$ and $L^{\Psi}(1 \mid (1,2)) = g P_{\theta,T}(A^c)$ when $\theta = c$ for some constant $g > 0$.

So $L^{\Psi}(\cdot \mid (1,2))$ is not a likelihood.

3. (a) (10 marks) Suppose that $x = (x_1, \ldots, x_n)$ is a sample from a $N(\mu_0, \sigma^2)$ distribution where μ_0 is known and $\sigma^2 > 0$ is unknown. Determine a minimal sufficient statistic for this model and justify your answer.

$L(\sigma^2 \mid x_1, \ldots, x_n) = (\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2\}$

and so by the Factorization Thm $T(x) = \sum_{i=1}^{n} (x_i - \mu_0)^2$ is sufficient. Also $\log L(\sigma^2 \mid x_1, \ldots, x_n)$

$= -\frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2$, $S(\sigma^2 \mid x_1, \ldots, x_n)$

$= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu_0)^2$ and solving

$S(\sigma^2 \mid x_1, \ldots, x_n) = 0$ give $\hat{\sigma}^2_{MLE}(x_1, \ldots, x_n)$

$= \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2$. Since we can compute $\sum_{i=1}^{n} (x_i - \mu_0)^2$ and it is sufficient, this proves it is minimal sufficient.

(b) (10 marks) For the situation described in (a) determine the UMVU estimator of σ and justify your answer.

We have that $w = \sum_{i=1}^{n} \frac{(x_i - \mu_0)^2}{\sigma^2} \sim$ chi-squared (n)

and so $E(w^{1/2}) = \int_0^\infty w^{1/2} \frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} w^{\frac{n}{2}-1} e^{-\frac{w}{2}} dw$

$= \frac{(\frac{1}{2})^{-\frac{1}{2}} \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \int_0^\infty \frac{(\frac{1}{2})^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} w^{\frac{n+1}{2}-1} e^{-w/2} dw$

$= 1$ since integral of chi-squared$(n+1)$ density.

Therefore, $s(x) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} (\frac{1}{2}) \sqrt{\sum_{i=1}^{n} (x_i - \mu_0)^2}$ satisfies

$E_{\sigma^2}(s(x)) = \sigma \quad \forall \sigma$ and $s(x)$ is unbiased. Since $s(x)$ is a function of the complete minimal statistic it is UMVU.

4. Suppose that $x = (x_1, \ldots, x_n)$ is a sample from a $N(\mu, \sigma_0^2)$ distribution where $\mu \in R^1$ is unknown and σ_0^2 is known.

(a) (5 marks) Suppose that we want to test $H_0 : \mu = 0$ when $n = 10^5, \sigma_0^2 = 1, \bar{x} = 0.01$ so the p-value $2(1 - \Phi(\sqrt{n}|\bar{x}|)) = 0.002$. What conclusions do you draw from this and mention any cautions that you think are appropriate in interpreting your answer?

Based on the p-value there is evidence that H_0 is false. Given the size of the sample, however, it may be that a deviation from H_0 has been detected that is of no practical importance. The caution is to determine whether or not a difference of 0.01 matters in the application.

(b) (5 marks) For the context discussed in (a), suppose, for different data, the p-value was determined to be equal to 0.40. Is this evidence for or against $H_0 : \mu = 0$? Justify your answer.

The best that can be said is that there is no evidence against H_0. We cannot interpret 0.40 as evidence in favor of H_0 because the p-value is uniformly distributed when H_0 is true so any value in $[0,1]$ can be obtained.

(c) (5 marks) An investigator collects a sample of size $n = 20$ where they are interested in assessing the hypothesis $H_0 : \mu = 0$ for the context discussed in (a) and obtains a p-value equal to 0.06. Since this is close to 0.05, where a p-value less than or equal to 0.05 is required for publication of their results in a specific journal, they decide to collect a further sample of 20 and for the combined data set obtain a p-value equl to 0.03. Explain why these results are not publishable in the journal.

The actual p-value equals

$$P(\text{1st test gives } \sqrt{20}|\bar{x}_1| > z_{0.975}) +$$

$$P(\text{2nd test gives } \sqrt{40}|\bar{x}_2| > z_{0.975} \mid \sqrt{20}|\bar{x}_1| \le z_{0.975}) \times P(\sqrt{20}|\bar{x}_1| \le z_{0.975})$$

$= .05 + \epsilon$ where $\epsilon > 0$ and so the p-value can never be smaller than 0.05.

(d) (5 marks) Provide a 0.95-confidence interval for μ in (a) and discuss the interpretation of this interval.

The 0.95 CI is given by

$$\bar{x} \pm \frac{1}{\sqrt{10^5}} z_{0.975} = 0.01 \pm \frac{1}{\sqrt{10^5}} 1.96 = 0.01 \pm 0.006.$$

The interpretation is as follows: conceive of repeating the sampling over and over again and each time compute the above interval, then 95% of the intervals so constructed will contain the true value of μ.

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(e) (5 marks) Explain what is meant by the power function of a test and show how you would calculate the power function of the test in (a).

The power function of a test φ is $E_\theta(\varphi)$ which is the probability of rejecting H_0 when θ is the true value. For the test in (a) the power function is given by

$$P_\mu\left(\sqrt{10^5}\,|\bar{X}| > z_{0.975}\right) = P_\mu\left(-z_{0.975} \le \sqrt{10^5}\,\bar{X} \le z_{0.975}\right)$$

$$= P_\mu\left(-z_{0.975} - \sqrt{10^5}\,\mu \le \sqrt{10^5}(\bar{X} - \mu) \le z_{0.975} - \sqrt{10^5}\,\mu\right)$$

$$= P_0\left(-z_{0.975} - \sqrt{10^5}\,\mu \le Z \le z_{0.975} - \sqrt{10^5}\,\mu\right)$$

where $Z \sim N(0,1)$

$$= \Phi\left(z_{0.975} - \sqrt{10^5}\,\mu\right) - \Phi\left(-z_{0.975} - \sqrt{10^5}\,\mu\right)$$

5. Suppose that the statistical model is given by

	$f_\theta(1)$	$f_\theta(2)$
$\theta = a$	$1/3$	$2/3$
$\theta = b$	$1/2$	$1/2$

and that the prior is given by $\pi(a) = 1/3, \pi(b) = 2/3$ and we observe the sample $(x_1, x_2, x_3) = (1, 1, 2)$.

(a) (5 marks) Determine the posterior distribution of θ.

$$\pi(\theta \mid 1,1,2) = \frac{\pi(\theta) f_\theta^2(1) f_\theta(2)}{(\frac{1}{3})(\frac{1}{3})^2(\frac{2}{3}) + (\frac{2}{3})(\frac{1}{2})^3} = \frac{\pi(\theta) f_\theta^2(1) f_\theta(2)}{\frac{2}{3}[(\frac{1}{3})^3 + (\frac{1}{2})^3]}$$

$$= \begin{cases} (1/3)^3 / [(\frac{1}{3})^3 + (\frac{1}{2})^3], & \theta = a. \\ (1/2)^3 / [(\frac{1}{3})^3 + (\frac{1}{2})^3], & \theta = b \end{cases}$$

(b) (5 marks) Determine the estimate of θ based on the Bayes rule for estimating θ when using the loss function $L(\theta, \psi) = 1$ when $\theta \neq \psi$ and $L(\theta, \psi) = 0$ otherwise.

We want $\delta(1,1,2) \in \{a, b\}$ that minimize

$$r(\delta \mid 1,1,2) = \pi(a \mid 1,1,2) L(a, \delta(1,1,2)) + \pi(b \mid 1,1,2) L(b, \delta(1,1,2))$$

$$= \begin{cases} \pi(a \mid 1,1,2) & \text{when } \delta(1,1,2) = b \\ \pi(b \mid 1,1,2) & \text{when } \delta(1,1,2) = a. \end{cases}$$

So set $\delta(1,1,2) = b$ since $\pi(a \mid 1,1,2) < \pi(b \mid 1,1,2)$

(c) (Bonus 5 marks) Determine the relative belief estimate of θ.

The relative belief estimate maximizes

$$RB(\theta \mid 1,1,2) = \frac{\pi(\theta \mid 1,1,2)}{\pi(\theta)}$$

$$= \begin{cases} (1/3)^2 / [\ ] & \text{when } \theta = a \\ 3(1/2)^4 / [\ ] & \text{when } \theta = b \end{cases}$$

so $\theta_{RB}(1,1,2) = b$ since $(1/3)^2 < 3(1/2)^4$.