8.1.8 The likelihood function is given by $L\left(x_1,\ldots,x_n \,|\, \sigma^2\right) =$ $\sigma^{-2n}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n\left(x_i-\mu_0\right)^2\right\}$. By factorization (Theorem 6.1.1), $\sum_{i=1}^n\left(x_i-\mu_0\right)^2$ is sufficient. Further, $E_{\sigma^2}\left(\sum_{i=1}^n\left(x_i-\mu_0\right)^2\right) = n\sigma^2$, so $n^{-1}\sum_{i=1}^n\left(x_i-\mu_0\right)^2$ is unbiased for σ^2. Since this sufficient statistic is complete, we have that $n^{-1}\sum_{i=1}^n\left(x_i-\mu_0\right)^2$ is UMVU for σ^2.

8.1.12 The likelihood function is given by $L\left(x_1,\ldots,x_n \,|\, \theta\right) = \theta^{-n}$ whenever $\theta > x_{(n)}$ and 0 otherwise. Therefore, when we know $x_{(n)}$ we know the likelihood function and so $x_{(n)}$ is sufficient. Then $x_{(n)}$ has density given by $\theta^{-n}nx^{n-1}$ for $0 < x < \theta$ and $E\left(x_{(n)}\right) = \int_0^\theta \theta^{-n}nx^n\,dx = \frac{\theta^{-n}n}{n+1}\,x^{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta$. So $\frac{n+1}{n}x_{(n)}$ is UMVU for θ.

8.1.16 We have that

$$\begin{aligned}\mathrm{MSE}_{(\mu,\sigma^2)}\left(cs^2\right) &= E_{(\mu,\sigma^2)}\left(\left(cs^2-\sigma^2\right)^2\right) = \sigma^4E_{(\mu,\sigma^2)}\left(\left(\frac{c}{(n-1)}X-1\right)^2\right)\\ &= \sigma^4\left\{\left(\frac{c}{n-1}\right)^2E_{(\mu,\sigma^2)}\left(X^2\right)-\frac{2c}{n-1}E_{(\mu,\sigma^2)}\left(X\right)+1\right\}\end{aligned}$$

where $X = (n-1)\,s^2/\sigma^2 \sim \chi^2(n-1)$. So $E_{(\mu,\sigma^2)}\left(X\right) = n-1$ and $\mathrm{Var}_{(\mu,\sigma^2)}\left(X\right) = 2(n-1)$, which implies $E_{(\mu,\sigma^2)}\left(X^2\right) = 2(n-1)+(n-1)^2$. Differentiating the above expression with respect to c, and setting the derivative equal to 0, gives that the optimal value satisfies

$$\frac{c}{(n-1)^2}E_{(\mu,\sigma^2)}\left(X^2\right)-\frac{1}{n-1}E_{(\mu,\sigma^2)}\left(X\right) = 0$$

or

$$c = (n-1)\,\frac{E_{(\mu,\sigma^2)}\left(X\right)}{E_{(\mu,\sigma^2)}\left(X^2\right)} = (n-1)\,\frac{(n-1)}{2(n-1)+(n-1)^2} = \frac{n-1}{n+1}.$$

We have that the bias equals

$$E_{(\mu,\sigma^2)}\left(cs^2\right)-\sigma^2 = (c-1)\,\sigma^2 = \left(\frac{n-1}{n+1}-1\right)\sigma^2 = \frac{-2\sigma^2}{n+1}\sigma^2.$$

8.1.22 The log-likelihood function is given by $l\left(\beta \,|\, x_1,\ldots,x_n\right) = n\alpha_0\ln\beta - \beta n\bar{x}$, so $S\left(\beta \,|\, x_1,\ldots,x_n\right) = n\alpha_0/\beta - n\bar{x}$, $S'\left(\beta \,|\, x_1,\ldots,x_n\right) = -n\alpha_0/\beta^2$, which implies $I\left(\beta\right) = n\alpha_0/\beta^2$. Since $\psi\left(\beta\right) = \beta^{-1}$, $\psi'\left(\beta\right) = -\beta^{-2}$ the information lower bound for unbiased estimators is given by $\left(1/\beta^4\right)\left(\beta^2/n\alpha_0\right) = 1/n\alpha_0\beta^2$. Note that by Exercise 8.1.7 $\bar{x}/\alpha_0$ is UMVU for β^{-1} and this has variance $\alpha_0/n\alpha_0^2\beta^2 = 1/n\alpha_0\beta^2$, which is the Cramer-Rao lower bound.

8.2.8 What we care in optimal hypothesis testing theory is type I and II errors, i.e., significance level and power function. Hence, we must ignore the difference of two test procedures whenever two tests have the same significance level and the same power function.

8.2.13 We have that $E_\theta(\varphi) = \alpha$ for every θ, so it is of exact size α. For this test, no matter what data is obtained, we randomly decide to reject H_0 with probability α.

8.2.14 Suppose that φ_0 is a size α UMP test for a specific problem and let φ be the test function of Problem 8.2.13. Then for θ such that the alternative is true we have that $E_\theta(\varphi_0) \geq E_\theta(\varphi) = \alpha$, so φ_0 is unbiased.

8.2.20 The UMP size α test for $H_0 : \lambda = \lambda_0$ versus $H_0 : \lambda = \lambda_1$ is of the form

$$\frac{L(\lambda_1 \mid x_1, \ldots, x_n)}{L(\lambda_0 \mid x_1, \ldots, x_n)} = \frac{(\lambda_1)^{n\bar{x}} e^{-\lambda_1}}{(\lambda_0)^{n\bar{x}} e^{-\lambda_0}} > c_0$$

or, equivalently, whenever $n\bar{x}(\ln\lambda_1 - \ln\lambda_0) > (\lambda_1 - \lambda_0)\ln c_0$, and since $\lambda_1 > \lambda_0$, this is equivalent to rejecting whenever $n\bar{x} > (\lambda_1 - \lambda_0)\ln c_0 / (\ln\lambda_1 - \ln\lambda_0)$. Now recall that $n\bar{x} \sim$ Poisson$(n\lambda_0)$ under H_0 so we must determine the smallest k such that $P_{\lambda_0}(n\bar{x} > k) \leq \alpha$ and then put $\gamma = (\alpha - P_{\lambda_0}(n\bar{x} > k)) / P_{\lambda_0}(n\bar{x} = k)$. Since this test does not involve λ_1, it is UMP size α for $H_0 : \lambda = \lambda_0$ versus $H_0 : \lambda > \lambda_0$. From Problem 8.2.19 we have that $P_\lambda(n\bar{x} > k) \leq 1 - \frac{1}{x!}\int_\lambda^\infty y^x e^{-y}\, dy$, and we see that this is increasing in λ. Therefore, this test is UMP size α for $H_0 : \lambda \leq \lambda_0$ versus $H_0 : \lambda > \lambda_0$.

8.2.21 When $H_0 : \mu = \mu_0$ holds, the log-likelihood and score functions are given by $l(x_1, \ldots, x_n \mid \sigma^2) = -n\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (x - \mu_0)^2$, $S(x_1, \ldots, x_n \mid \sigma^2) = -\frac{n}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (x - \mu_0)^2$. Then $S(x_1, \ldots, x_n \mid \sigma^2) = 0$ leads to the MLE $\hat{\mu}_{H_0} = \mu_0$, $\hat{\sigma}^2_{H_0} = \frac{1}{n}\sum_{i=1}^n (x - \mu_0)^2$, and the maximized log-likelihood equals $l(x_1, \ldots, x_n \mid \hat{\sigma}^2_{H_0}) = n\ln n - n\ln\sum_{i=1}^n (x - \mu_0)^2 - \frac{n}{2}$.

When $H_a : \mu \neq \mu_0$ holds, the log-likelihood function is given by $l(x_1, \ldots, x_n \mid \mu, \sigma^2) = -n\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (x - \mu)^2$.

By Example 6.2.6 the MLE is given by $\hat{\mu} = \bar{x}, \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x - \bar{x})^2$ and the maximized log-likelihood is given by $l(x_1, \ldots, x_n \mid \hat{\mu}, \hat{\sigma}^2) = n\ln n - n\ln\sum_{i=1}^n (x - \bar{x})^2 - \frac{n}{2}$. Then the likelihood ratio test rejects whenever

$$\begin{aligned} &2\left(l\left(x_1, \ldots, x_n \mid \hat{\mu}, \hat{\sigma}^2\right) - l\left(x_1, \ldots, x_n \mid \hat{\sigma}^2_{H_0}\right)\right) \\ &= 2\left(-n\ln\sum_{i=1}^n (x - \bar{x})^2 + n\ln\sum_{i=1}^n (x - \mu_0)^2\right) = 2n\ln\frac{\sum_{i=1}^n (x - \mu_0)^2}{\sum_{i=1}^n (x - \bar{x})^2} > x_{1-\alpha} \end{aligned}$$

where $x_{1-\alpha}$ is the $(1-\alpha)$th quantile of the $\chi^2(2-1) = \chi^2(1)$ distribution.