8.4.8 Suppose we have that $\delta(s,\cdot)$ is degenerate at $d(s)$ for each s. Then clearly $d : S \to \mathcal{A}$.

Now suppose we have $d : S \to \mathcal{A}$ and define

$$\delta(s,B) = \begin{cases} 1 & d(s) \in B \\ 0 & \text{otherwise} \end{cases}$$

for $B \subset \mathcal{A}$. Then $\delta(s,\mathcal{A}) = 1$ and, if $B_1, B_2, \ldots$ are mutually disjoint subsets of $\mathcal{A}$, then $d(s) \in B_i$ for one i (and only one) if and only if $d(s) \in \cup_{j=1}^{\infty} B_j$, so $\delta\left(s, \cup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \delta(s, B_j)$. Therefore, $\delta(s,\cdot)$ is a probability measure for each s and δ is a decision function.

Now, using the fact that $\delta(s,\cdot)$ is a discrete probability measure degenerate at $d(s)$, we have that $R_\delta(\theta) = E_\theta\left(E_{\delta(s,\cdot)}(L(\theta,a))\right) =$
$E_\theta(\delta(s,\{d(s)\})(L(\theta,d(s))) = E_\theta(L(\theta,d(s)))$ since $\delta(s,\{d(s)\}) = 1$.

8.4.9

(a) Consider the decision function $d_{\theta_0}(s) \equiv A(\theta_0)$. Then note that $R_{d_{\theta_0}}(\theta_0) = 0$. Then, if δ is optimal, we must have that $R_\delta(\theta_0) \le R_{d_{\theta_0}}(\theta_0)$ for every θ_0, so $R_\delta(\theta) \equiv 0$. But this implies that $E_{\delta(s,\cdot)}(L(\theta,a)) = 0$ at every s, where $P_\theta(\{s\}) > 0$. Since $L(\theta,a) \ge 0$, then Challenge 3.3.29 implies that $\delta(s,\{L(\theta,a) = 0\}) = 1$ and, since $L(\theta,a) = 0$ if and only if $a = A(\theta)$, this implies that $\delta(s,\cdot)$ is degenerate at $A(\theta)$ for each s for which $P_\theta(\{s\}) > 0$.

(b) Part (a) proved that, for an optimal δ, $\delta(s,\cdot)$ is degenerate at $A(\theta)$ for each s for which $P_\theta(\{s\}) > 0$. But if there exists s such that $P_{\theta_1}(\{s\}) > 0$ and $P_{\theta_2}(\{s\}) > 0$ and $A(\theta_1) \ne A(\theta_2)$, then this cannot happen and so no optimal δ can exist.

8.4.10 Suppose δ is not minimax. Then there exists decision function δ^* such that $\sup_\theta R_{\delta^*}(\theta) < \sup_\theta R_\delta(\theta)$. But since $R_\delta(\theta)$ is constant in θ this implies that $R_{\delta^*}(\theta) < R_\delta(\theta)$ for every θ and so δ is not admissible, contradicting the hypothesis. Therefore, δ must be minimax.