1\. (a) $L_{\Psi}(1|1) = \frac{1}{2}$ $\quad L(2|1) = \frac{1}{3}$

$L_{\Psi}(1|2) = \frac{1}{3}$ $\quad L(2|2) = \frac{4}{5}$

$L_{\Psi}(1|3) = \frac{1}{2}$ $\quad L(2|3) = \frac{1}{6}$

and any positive multiple of these give all possible profile likelihoods.

(b) $C_{1/2}(1) = \{\psi : L_{\Psi}(\psi|1) \geq \frac{1}{2}\sup_{\psi} L(\psi|1)\}$

$= \{\psi : L_{\Psi}(\psi|1) \geq 1/4\} = \{1, 2\}$

$C_{1/2}(2) = \{\psi : L_{\Psi}(\psi|2) \geq \frac{2}{5}\} = \{2\}$

$C_{1/2}(3) = \{\psi : L_{\Psi}(\psi|3) \geq \frac{1}{4}\} = \{1\}$

2\. (a) $L(\mu, \sigma^2|x) = (\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2 - \frac{(n-1)s_x^2}{2\sigma^2}\}$

and $L(\psi, \sigma^2|x) = (\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{n}{2\sigma^2}(\bar{x}-\frac{\sigma}{\psi})^2 - \frac{(n-1)s_x^2}{2\sigma^2}\}$

and note that $\psi \in \mathbb{R} \setminus 0$, $\sigma^2 > 0$ are unconstrained so $L_{\Psi}(\psi|x) = \sup_{\sigma^2>0} L(\psi, \sigma^2|x)$ and we need to maximize $L(\psi, \sigma^2|x)$ as a function of $\sigma^2$.

(b) Taking the log of the likelihood we obtain

$l(\psi, \sigma^2|x) = -\frac{n}{2}\log\sigma^2 - \frac{n}{2\sigma^2}(\bar{x}-\frac{\sigma}{\psi})^2 - \frac{(n-1)s_x^2}{2\sigma^2}$ and

$$\frac{\partial l(\psi,\sigma^2|x)}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{n}{2\sigma^4}\left(\bar{x} - \frac{(\sigma^2)^{1/2}}{\psi}\right)^2$$

$$+ \frac{n}{2\sigma^2}\, 2\left(\bar{x} - \frac{(\sigma^2)^{1/2}}{\psi}\right)\frac{1}{2}\frac{(\sigma^2)^{-\frac{1}{2}}}{\psi} + \frac{(n-1)s_x^2}{2\sigma^4}$$

Then setting this equal to 0 and multiplying through by $-2\sigma^5/n$ we get

$$\sigma^3 - \sigma(\bar{x}^2 - 2\bar{x}\sigma/\psi + \sigma^2/\psi^2) - \sigma^3(\bar{x} - \sigma/\psi)\frac{\sigma^{-1}}{\psi} - \frac{n-1}{n}s_x^2\sigma$$

$$= \sigma^3 - \bar{x}^2\sigma + \frac{2\bar{x}}{\psi}\sigma^2 - \frac{\sigma^3}{\psi^2} - \frac{\bar{x}}{\psi}\sigma^2 + \frac{\sigma^3}{\psi^2} - \frac{n-1}{n}s_x\sigma$$

$$= \sigma^3 + \frac{\bar{x}}{\psi}\sigma^2 - \left(\bar{x}^2 + \frac{n-1}{n}s_x\right)\sigma = 0$$

or $\sigma^2 + \frac{\bar{x}}{\psi}\sigma - \left(\bar{x}^2 + \frac{n-1}{n}s_x\right) = 0$

so
$$\sigma = \frac{-\frac{\bar{x}}{\psi} \pm \sqrt{\frac{\bar{x}^2}{\psi^2} + 4\left(\bar{x}^2 + \frac{n-1}{n}s_x^2\right)}}{2}$$

$$= \frac{1}{2}\left(\frac{-\bar{x}}{\psi} \pm \sqrt{\left(\frac{\bar{x}}{\psi}\right)^2 + \frac{4}{n}\sum_{i=1}^{n} x_i^2}\right) \text{ since } \bar{x}^2 + \frac{n-1}{n}s_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

and since $\sigma$ is not negative the root must be

$$\sigma(\psi) = \frac{1}{2}\left(-\frac{\bar{x}}{\psi} + \sqrt{\frac{\bar{x}^2}{\psi^2} + \frac{4}{n}\sum_{i=1}^{n} x_i^2}\right)$$

Then the profile likelihood fn is

$$L_{\Psi}(\psi|x) = (\sigma^2(\psi))^{-\frac{n}{2}}\exp\left\{-\frac{n}{2\sigma^2(\psi)}\left(\bar{x} - \frac{\sigma(\psi)}{\psi}\right)^2 - \frac{(n-1)s_x^2}{\sigma^2(\psi)}\right\}.$$