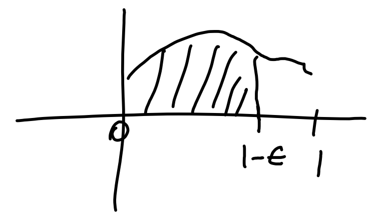
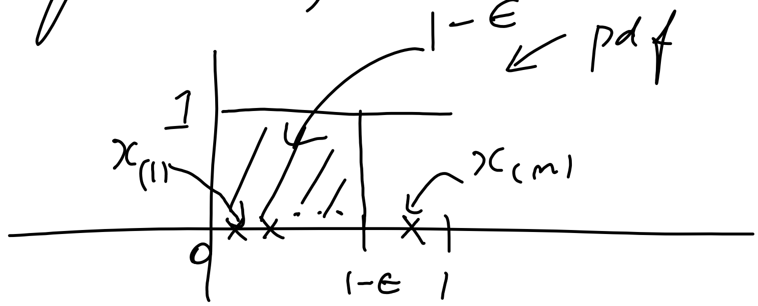
Week 11



eg uniform $(0,1)$



Let $X_1, \ldots, X_n$ be iid uniform $(0,1)$

$$\underbrace{X_{(1)} < X_{(2)} < \cdots < X_{(n)}}_{\text{order statistics}}$$

We want to show $X_{(n)} \xrightarrow{?} 1$ as $n \to \infty$

Convergence types

$Y_1, Y_2, \ldots \; ; \; Y$

sample space

pointwise convergence $\quad Y_n(s) \to Y(s), \quad \forall s \in S$

Notation $\quad Y_n \longrightarrow Y$

Convergence wp1

$Y_n \xrightarrow{wp1} Y$ if $P(Y_n \to Y) = 1$

(ie the set of sample points where it doesn't converge has prob 0)

convergence in mean square

$Y_n \xrightarrow{ms} Y$ if $E(Y_n - Y)^2 \to 0$

Note Let $\epsilon > 0$ & suppose we know $Y_n \xrightarrow{ms} Y$

$$P(|Y_n - Y| > \epsilon) = P\left[(Y_n - Y)^2 > \epsilon^2\right]$$

$$\leq \frac{E(Y_n - Y)^2}{\epsilon^2} \to 0$$

Convergence in probability

$Y_n \xrightarrow{P} Y$ if $\forall \epsilon > 0$

$$P(|Y_n - Y| \leq \epsilon) \to 1$$

Back to the uniform (0,1)

$X_{(n)} \xrightarrow{P} 1$ $\quad \left\{ X_{(n-\underset{\substack{\uparrow \\ \text{fixed}}}{k})} \xrightarrow{P} 1 \right\}$

Let $\epsilon > 0$. Then

$$
\begin{aligned}
P(|X_{(n)} - 1| > \epsilon) &= P(1 - X_{(n)} > \epsilon) \\
&= P(X_{(n)} < 1 - \epsilon) \\
&= P(X_1 < 1-\epsilon, \dots, X_n < 1-\epsilon) \\
&= P(X_1 < 1-\epsilon)\, P(X_2 < 1-\epsilon) \cdots P(X_n < 1-\epsilon) \\
&= \left[P(X_1 < 1-\epsilon) \right]^n \\
&= (\underbrace{1-\epsilon}_{0<1-\epsilon<1})^n \longrightarrow 0
\end{aligned}
$$

$\therefore \quad X_{(n)} \xrightarrow{P} 1$

In the same way $X_{(1)} \xrightarrow{P} 0$. (Show it)

How fast does $X_{(n)} \to 1$?

Look at

$$Y_n = n(1 - X_{(n)})$$

$$P(Y_n \le \underset{\geq 0}{y}) = P(n(1 - X_{(n)}) \le y)$$

$$= P(1 - X_{(n)} \le y/n)$$

$$= P\left(X_{(n)} \geq 1 - \frac{y}{n}\right)$$

$$= 1 - P\left(X_{(n)} < 1 - \frac{y}{n}\right)$$

$$= 1 - \left(1 - \frac{y}{n}\right)^n$$

$$\to 1 - e^{-y}$$

which is the df of an exponential(1).

This says

$$n[1 - X_{(n)}] \overset{d}{\approx} \text{exponential}(1)$$

Convergence in dist'n

$X_n \xrightarrow{d} X$ $\{ X_n \overset{d}{\approx} X \}$

if

$$P(X_n \le x) \longrightarrow P(X \le x), \quad \forall x\text{'s st } \underbrace{P(X=x)=0}_{\text{no jump}}$$

$\xrightarrow{d}$

df's mgf's $\leftarrow$ $m(t) = E(e^{tX})$

pf's cf's $\leftarrow$ $c(t) = E(e^{\underbrace{i}_{\sqrt{-1}} tX})$

pdf's pgf's

If $m(t)$ exists around $t=0$ then

$$c(t) = m(it)$$

eg $X \sim N(0,1) \Rightarrow m(t) = e^{t^2/2}$

$$\Rightarrow c(t) = e^{-t^2/2}$$

Weak Law of Large #'s

Let $X_1, X_2, \dots$ be iid with mean μ & variance σ^2. Then

$$\overline{X} = \frac{X_1 + \dots + X_n}{n} \xrightarrow{P} \mu$$

($\xrightarrow{wP1}$ Strong Law)

Proof: Look at

$$E(\overline{X} - \mu)^2 = Var(\overline{X}), \quad \because E(\overline{X}) = \mu$$

$$= \frac{\sigma^2}{n} \to 0$$

$$\Rightarrow \overline{X} \xrightarrow{ms} \mu \Rightarrow \overline{X} \xrightarrow{P} \mu$$

Central Limit Theorem

Let $X_1, X_2, \dots$ be iid with mean μ, variance σ^2 {& mgf $m(t)$}.

Then

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1)$$

Proof: Already done.

Back to Order Statistics

$X_1, X_2, \cdots, X_n$ iid, pdf f, df F

$$X_{(1)} < \cdots < X_{(n)}$$

order stats

pdf of $X_{(r)}$ — call it $f_{(r)}$

— df $F_{(r)}$

— tail f'n $\bar{F}_{(r)} = 1 - F_{(r)}$

"Easy" cases $r = 1, n$

$r = n$

$$F_{(n)}(x) = P(X_{(n)} \leq x)$$

$$= P(X_1 \le x, X_2 \le x, \dots, X_n \le x)$$

$$= P(X_1 \le x)\, P(X_2 \le x) \cdots P(X_n \le x)$$

$$= [F(x)]^n$$

$$\Rightarrow f_{(n)}(x) = n\, F(x)^{n-1} f(x)$$

$\underline{\underline{r=1}}$

$$\bar{F}_{(1)}(x) = P(X_{(1)} > x)$$

$$= P(X_1 > x, \dots, X_n > x)$$

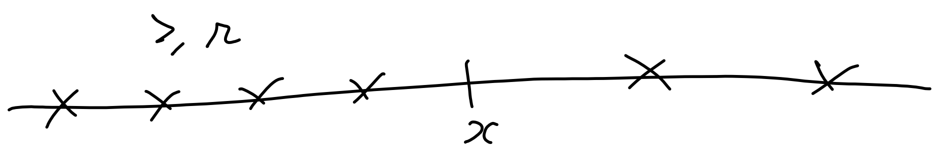
$$= [\bar{F}(x)]^n \qquad \{\bar{F}(x) = P(X_k > x)\}$$

$$\Rightarrow f_{(1)}(x) = -\bar{F}_{(1)}'(x) = n[\bar{F}(x)]^{n-1} f(x)$$

$$= n[1-F(x)]^{n-1} f(x)$$

$\underline{\underline{r}}$

$$f_{(r)}(x) = \frac{d}{dx} F_{(r)}(x) = \frac{d}{dx} P(X_{(r)} \le x)$$



of the original X's which are $\leq x \sim \text{binomial}(n, F(x))$

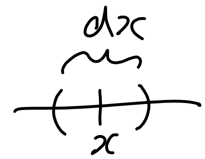
$$\{X_{(r)} \leq x\} = \{\# \text{ of } X\text{'s} \leq x \text{ is} \geq r\}$$

$$= \sum_{k=r}^{n} \binom{n}{k} F(x)^k \bar{F}(x)^{n-k}$$

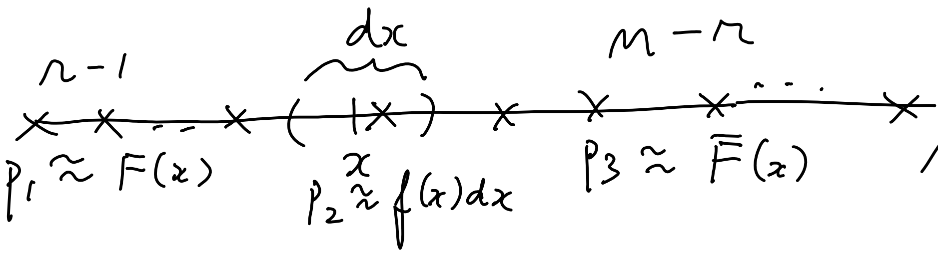
+ so

$$f_{(r)}(x) = \frac{d}{dx} \quad '' \quad = \text{ do it}$$

<u>or</u>

$f_{(r)}(x)\,dx \approx P\Big(X_{(r)}$ is in 

$\Big)$

$\approx P\Big($ 

$\Big)$

$$= \binom{n}{r-1,\ 1,\ n-r} F(x)^{r-1} f(x)dx\ \bar{F}(x)^{n-r}$$

$$\text{“}\Rightarrow\text{”} \quad f_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} \bar{F}(x)^{n-r} f(x)$$

Let $r_1 < r_2$ & suppose we want the joint pdf of $X_{(r_1)}$ & $X_{(r_2)}$. Call

$$f_{(r_1)(r_2)}(x, y) \quad (= 0 \text{ for } x > y)$$

If $x < y$

$$f_{(r_1)(r_2)}(x,y)\,dx\,dy$$

$$\approx P\Big(X_{(r_1)} \text{ is in } \underset{x}{\overset{dx}{(\,|\,)}} \;\&\; X_{(r_2)} \text{ is in } \underset{y}{\overset{dy}{(\,|\,)}} \Big)$$

$$\approx P\Big(\underbrace{\overset{r_1-1}{\times \times \times \cdots \times}}_{p_1 \approx F(x)} \underbrace{\overset{dx}{(\,|\!\times)}}_{\substack{x \\ p_2 \approx f(x)dx}} \underbrace{\overset{r_2-1-r_1}{\times \times \cdots \times}}_{p_3 \approx F(y)-F(x)} \underbrace{\overset{dy}{(\,|\!\times)}}_{\substack{y \\ p_4 \approx f(y)dy}} \underbrace{\overset{n-r_2}{\times \times \cdots \times}}_{p_5 \approx \bar{F}(y)} \Big)$$

$$\approx \binom{n}{r_1-1,\ 1,\ r_2-1-r_1,\ 1,\ n-r_2} F(x)^{r_1-1} f(x)\,dx \left[F(y)-F(x)\right]^{r_2-1-r_1} \times f(y)\,dy\ \bar{F}(y)^{n-r_2}$$

$$\longrightarrow f_{(r_1)(r_2)}(x,y)$$

The jt pdf of $X_{(1)}, \ldots, X_{(n)}$ is easier to get! Call it $f(x_{(1)}, \ldots, x_{(n)})$

$$0 \underset{x_{(1)}}{\overset{dx_{(1)}}{(\,|\!\times)}} \ 0 \ \underset{x_{(2)}}{\overset{dx_{(2)}}{(\,|\!\times)}} \ 0 \ \cdots \ \underset{x_{(n)}}{\overset{dx_{(n)}}{(\,|\!\times)}} \ 0$$

& So

$$f(x_{(1)},\dots,x_{(n)})\,dx_{(1)}\cdots dx_{(n)}$$

$$\approx \binom{n}{0,1,0,1,\dots,1,0} f(x_{(1)})\,dx_{(1)}\cdots f(x_{(n)})\,dx_{(n)}$$

$$\Rightarrow f(x_{(1)},\dots,x_{(n)}) = n!\,f(x_{(1)})\cdots f(x_{(n)}),\quad x_{(1)}<\dots<x_{(n)}$$

The End