Week #2

$(S, \{\text{events}\}, P)$ — probability space

$A \subset B$; $(A \Rightarrow B)$; $I_A \leq I_B$

Note $A \subset B \iff I_A \leq I_B$

Proof: Assume $A \subset B$. We must show

$$I_A(s) \leq I_B(s) \quad , \forall s \in S$$

So let $s \in A$. Then $I_A(s) = 1$. But $A \subset B$ & $\therefore s \in B$ & hence $I_B(s) = 1$. Consequently,

$$I_A(s) \leq I_B(s) \quad , \forall s \in A$$

Now take $s \in A^c$. Then $I_A(s) = 0$. Since $I_B(s) \geq 0$ no matter what s is we get

$$I_A(s) \leq I_B(s) \quad , \forall s \in A^c$$

$$\therefore \quad I_A \leq I_B$$

Now assume $I_A \leq I_B$. We must show $A \subset B$.

To see this let $s \in A$. We must show $s \in B$.
If it isn't then $I_B(s) = 0$. But $I_A(s) = 1$ & $I_A(s) \le I_B(s)$. This can't be so s must be in B. $\therefore$ $A \subset B$ & hence

$$A \subset B \iff I_A \le I_B$$

qed

Calculation using symmetry

eg Select 2 cards. P(both are spades)?

Sol'n # of "2 card hands"

$$= \binom{52}{2} = \frac{52!}{2!\,50!} = \frac{52 \times 51 \times \cancel{50!}}{2 \times 1 \times \cancel{50!}}$$

of "2 card hands" made up only of spades is $\binom{13}{2}$

$$\therefore \quad P(\text{both are spades}) = \binom{13}{2} \Big/ \binom{52}{2}$$

eg Roll a fair die & let X = # of dots.

$$P(X \text{ is even}) = 3/6$$

$$P(X \geq 4) = 3/6$$

Let $B = \{X \text{ is even}\}$ & $A = \{X \geq 4\}$

Suppose you are told A has occured then one would <u>update</u> the probability of B to $2/3$. This is the <u>conditional</u> probability of B given A. The usual notation is

$$P(B|A) = 2/3 \quad \text{here}$$

$$\{ P_A(B) \}$$

<u>Def'n</u> $$P(B|A) = \frac{P(AB)}{P(A)}$$

<u>Note</u>, For fixed A, $P(\cdot|A)$ satisfies the Laws of P.

2. $P(AB) = P(A)\, P(B|A)$

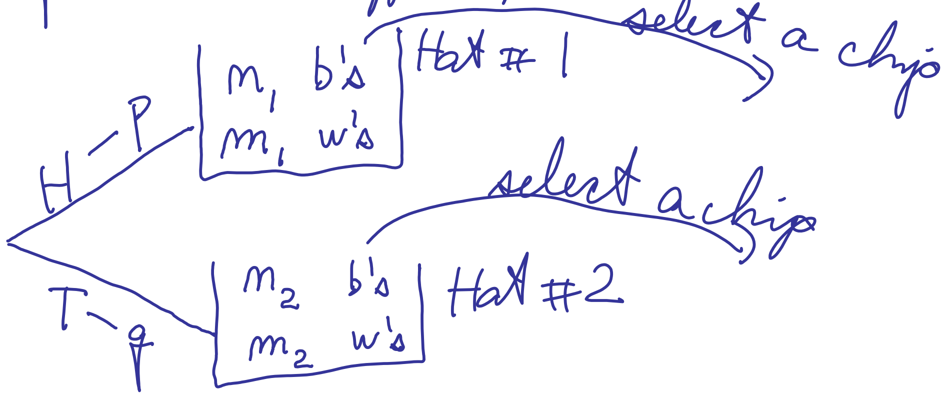
3. A + B independent $\left(P(AB) = P(A)\, P(B) \right)$
gives us $P(B|A) = P(B)$.

eg Back to P(both are spades)?

Sol'n Let $A = \{$spade on 1st draw$\}$
$B = \{$spade on 2nd draw$\}$

$$P(AB) = P(\text{both are spades})$$
$$= P(A)\, P(B|A)$$
$$= \frac{13}{52} \times \frac{12}{51}$$

Bayes formula type problems



– P(black chip) ?

– P(H | b) ?

Let N_1 = # of chips in Hat 1

+ N_2 = # of chips in Hat 2

<u>Sol'n</u> $P(b) = P(b \text{ and } H) + P(b \text{ and } T)$

$$= P(H)\,P(b \mid H) + P(T)\,P(b \mid T)$$

$$= p \times \frac{m_1}{N_1} + q \times \frac{m_2}{N_2}$$

$$P(H \mid b) = \frac{P(H \text{ and } b)}{P(b)} = \frac{P(b \text{ and } H)\checkmark}{P(b)\checkmark}$$

<u>rv</u> $X : S \to \mathbb{R}$

Expectation, expected value, mean of X

$E(X)$

We must have

$$\boxed{E(I_A) = P(A)} \qquad (*)$$

We would like E to have the following properties:

- $E(1) = 1$ $\qquad$ (I)
- $X \geq 0 \Rightarrow E(X) \geq 0$ $\qquad$ (II)
- $E(cX + dY) = c\,E(X) + d\,E(Y)$ $\qquad$ (III)
- $E\left(\sum_{k=1}^{\infty} X_k\right) = \sum_{k=1}^{\infty} E(X_k)$ $\qquad$ (IV)

  $X_k \geq 0$

<u>Theorem</u> There exists a unique $E : \{rv\text{'s}\} \to \mathbb{R}$ satisfying $(*)$ + (I) → (IV).

We know

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

<u>Proposition</u> $E(X_1+X_2+\cdots+X_N) = E(X_1)+E(X_2)+\cdots+E(X_N)$,

$N \geq 2$

<u>Proof</u> We know the result is true for $N=2$. Assume it's true for $N = m \geq 2$. Now take $N = m+1$. Look at

$$E(X_1+X_2+\cdots+X_m+X_{m+1})$$

$$= E[(X_1+X_2+\cdots+X_m) + X_{m+1}]$$

$$= E(X_1+X_2+\cdots+X_m) + E(X_{m+1})$$

$$= E(X_1)+\cdots+E(X_m) + E(X_{m+1})$$

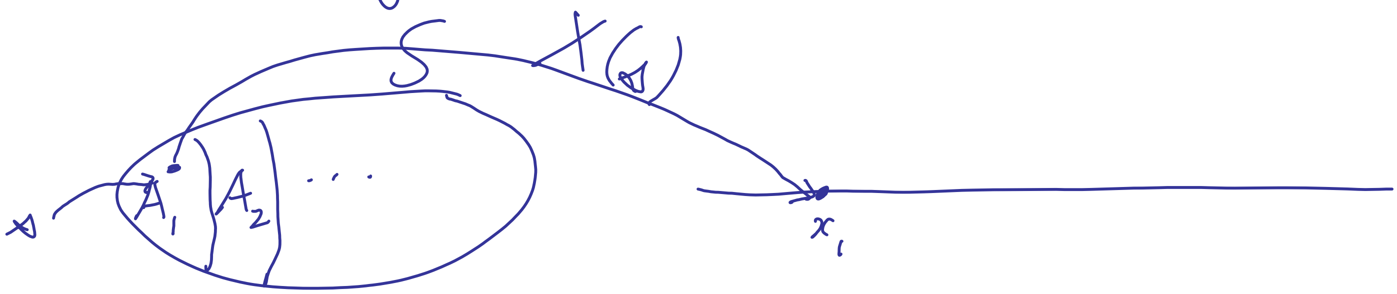
& so the result holds for $N = m+1$ & hence $\forall\ N \geq 2$ by induction.

<u>qed</u>

discrete rv's

A rv X is discrete if its range is countable. Suppose the range is the set $\{x_1, x_2, \dots\}$. Now let $A_k = \{X = x_k\}$

$A_1, A_2, \dots$ partition S.



$$\therefore \quad X = x_1 I_{A_1} + x_2 I_{A_2} + \cdots$$

&

$$g(X) = g(x_1) I_{A_1} + g(x_2) I_{A_2} + \cdots$$

"Hence"

$$E[g(X)] = g(x_1)\, E(I_{A_1}) + g(x_2)\, E(I_{A_2}) + \cdots$$

$$= g(x_1)\, P(X = x_1) + g(x_2)\, P(X = x_2) + \cdots$$

$$= \sum_{\text{all } x} g(x)\, P(X = x)$$

Call

$$f(x) = P(X=x)$$

the probability function (pf).

Note

$$\left.\begin{aligned} f(x) &\geq 0 \\ \sum_{\text{all } x} f(x) &= 1 \end{aligned}\right\}$$ conditions for a function to be a pf

$E(X)$ is also called <u>the mean</u> — μ

$E(X^2)$ is called the 2nd moment

$E[(X-\mu)^2]$ is the variance of X — σ^2, $Var(X)$

$SD(X) = \sqrt{Var(X)}$ — σ

<u>Note</u> $Var(X) = E[(X-\mu)^2]$

$$= E(X^2 + \mu^2 - 2\mu X)$$

$$= E(X^2) + \mu^2 - 2\mu \underbrace{E(X)}_{\mu} = E(X^2) - \mu^2$$