Week # 3

Recall A_1, A_2 are independent if

$$P(A_1 A_2) = P(A_1)\, P(A_2)$$

__Def'n__ $A_1, A_2, \dots$ are independent if for every $i_1 < i_2 < \dots < i_k$

$$(*) \qquad P(A_{i_1} A_{i_2} \cdots A_{i_k}) = P(A_{i_1})\, P(A_{i_2}) \cdots P(A_{i_k})$$

__Note__ $I_{A_1} I_{A_2} = I_{A_1 A_2}$

$$I_{A_{i_1} \cdots A_{i_k}} = I_{A_{i_1}} \cdots I_{A_{i_k}}$$

So $(*)$ is

$$E(I_{A_{i_1} \cdots A_{i_k}}) = E(I_{A_{i_1}} \cdots I_{A_{i_k}}) = E(I_{A_{i_1}}) \cdots E(I_{A_{i_k}})$$

__Def'n__ $X_1, X_2, \cdots$ are independent if every "X_1 event", "X_2 event", $\cdots$ are independent

__Theorem__ If $X_1, X_2, \cdots$ are independent then

$$E[h_1(X_1)\, h_2(X_2) \cdots] = E[h_1(X_1)]\, E[h_2(X_2)] \cdots, \quad \text{"}\forall\text{"}\ h_1, h_2, \cdots$$

Pf: Not here.

Let range of $X \subset \{0, 1, \cdots\}$. This is a special rv called a <u>counting</u> rv.

<u>eg</u> Let A be an event & I_A its indicator. This is a counting rv (possible values are 0 or 1). Suppose $p = P(A)$ & $q = 1-p$ & assume $0 < p < 1$.

$$\begin{cases} P(I_A = x) = p^x q^{1-x}, & x = 0, 1 \\ \quad\quad = 0, & \text{ow} \end{cases}$$

$$E(I_A) = 0 \times q + 1 \times p = p$$

$$E(\underbrace{I_A^k}_{I_A}) = 0^k \times q + 1^k \times p = p \qquad (k > 0 \text{ \& an integer})$$

$$\text{Var}(I_A) = E(I_A^2) - (E(I_A))^2 = p - p^2 = pq$$

<u>Note</u> $\text{Var}(X) = E[(X - \underset{\substack{\uparrow \\ E(X)}}{\mu})^2] = E(X^2) - \mu^2$

<u>Terminology</u> Call X a Bernoulli(p) rv if $X \in \{0, 1\}$ & $P(X = 1) = p$. $X \sim \text{Bernoulli}(p)$

Let $X_1, X_2, \cdots$ be iid Bernoulli (p)

identically
ind distributed

rv's.

Let Y = # of 1's until "Time" n.
$= X_1 + X_2 + \cdots + X_n$

Possible values for Y are $0, 1, \cdots, n$. Y is called a binomial(n, p) rv.

$$P(Y = k) \, ?$$

Sol'n #1 Count & use indendence (see text)

Sol'n #2 Use probability generating f'ns.

Def'n If Y is a counting rv its pgf is

$$G(s) = E(s^Y) = \sum_{k=0}^{\infty} s^k P(Y = k)$$

Notice $|G(s)| \leq \sum_{k=0}^{\infty} |s|^k P(Y=k) \leq \sum_{k=0}^{\infty} P(Y=k) = 1$,

if $|s| \leq 1$.

Δ inequality $(|a+b| \leq |a| + |b|)$

$$G(s) = P(Y=0) + P(Y=1)s + P(Y=2)s^2 + \cdots$$

Since a polynomial determines all the coefficients, knowing $G \Rightarrow$ know the probabilities.

$\boxed{\text{Application to the calculation of the binomial}(n,p)\text{ probabilities.}}$

$$Y \sim \text{binomial}(n,p)$$
$$= X_1 + \cdots + X_n$$

iid Bernoulli(p)

pgf of X_1

$$G_{X_1}(s) = E(s^{X_1}) = \underbrace{P(X_1=0)}_{q} + \underbrace{P(X_1=1)}_{p}s$$
$$= q + ps$$

pgf of Y

$$G_Y(s) = E(s^Y) = \sum_{k=0}^{n} P(Y=k)s^k$$
$$||$$

$$E\left(s^{X_1+\cdots+X_n}\right)$$

$$= E\left(s^{X_1} s^{X_2} \cdots s^{X_n}\right) = E\left(s^{X_1}\right) E\left(s^{X_2}\right) \cdots E\left(s^{X_n}\right)$$

$$= G_{X_1}(s)\, G_{X_2}(s) \cdots G_{X_n}(s)$$

Aside

Note The above shows that the pgf of a finite sum of ind rv's = the product of the pgf's.

$$= (q+ps)^n = \sum_{k=0}^{n} \binom{n}{k} (ps)^k q^{n-k}$$

$$= \sum_{k=0}^{n} \underbrace{\binom{n}{k} p^k q^{n-k}}_{P(Y=k)} s^k$$

So if $Y \sim \text{binomial}(n,p)$

probabilities $P(Y=k) = \binom{n}{k} p^k q^{n-k}$, $k = 0, \ldots, n$

pgf $G(s) = (q+ps)^n = (1-p+ps)^n = (1+p(s-1))^n$

$E(Y)$?

<u>Easiest way</u> $\quad Y = X_1 + \cdots + X_n$

$$\Rightarrow E(Y) = E(X_1) + \cdots + E(X_n) = np$$

<u>Another way</u>

$$E(Y) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} \overset{\text{do it}}{\downarrow}{=} np$$

<u>Var(Y)?</u> $= E(Y^2) - [E(Y)]^2$

<u>Method #1</u>

$$E(Y^2) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k} \overset{\text{do it}}{\downarrow}{=}$$

<u>Method #2</u> — use ~~pgf~~

$$G(s) = E(s^Y)$$

$$G^{(1)}(s) = E\left(\frac{d}{ds} s^Y\right) = E\left(Y s^{Y-1}\right)$$

$$G^{(2)}(s) = E\left(Y(Y-1) s^{Y-2}\right)$$

$$\therefore \quad E(Y) = G^{(1)}(1)$$

$$E[Y(Y-1)] = G^{(2)}(1)$$

So $\begin{cases} G^{(2)}(1) = E(Y^2) - E(Y) \\ G^{(1)}(1) = E(Y) \end{cases}$

$$\Longrightarrow E(Y^2) = G^{(1)}(1) + G^{(2)}(1)$$

Here

$$G(s) = (q+ps)^n$$

$$\Rightarrow G^{(1)}(s) = n(q+ps)^{n-1}p \overset{s=1}{=} np$$

$$G^{(2)}(s) = n(n-1)(q+ps)^{n-2}p^2 \overset{s=1}{=} n(n-1)p^2$$

$\therefore$ $E(Y^2) = n(n-1)p^2 + np$

& so

$$Var(Y) = n(n-1)p^2 + np - (np)^2 = np - np^2$$
$$= npq$$

$$\underbrace{SD(Y)}_{\sigma} = \sqrt{Var(Y)} = \sqrt{npq}$$

One last thing. If $Y \sim$ binomial(n, p)

$$G(s) = [1 + p(s-1)]^n$$
$$= \left(1 + \frac{\overset{\lambda}{\widetilde{np}}(s-1)}{n}\right)^n$$

$$= \left(1 + \frac{\lambda(s-1)}{n}\right)^n \approx e^{\lambda(s-1)},$$

$\left[\left(1+\frac{x}{n}\right)^n \to e^x \right]$ for large n & λ fixed.

What kind of rv has pgf $e^{\lambda(s-1)}$?

Call it X.

$$G_X(s) = e^{\lambda(s-1)}$$

$$= e^{-\lambda} e^{\lambda s}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!}$$

$$= \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} s^k$$

& so

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

These are the Poisson(λ) probabilities.

Back to iid Bernoulli(p) sequence

$X_1, X_2, \ldots$

Let Y = "time" until the first 1. This is a geometric(p) rv & clearly

$$P(Y=k) = P(X_1 = 0, X_2 = 0, \ldots, X_{k-1} = 0, X_k = 1)$$

means and (same as intersection)

$$= P(X_1=0)\,P(X_2=0)\cdots P(X_{k-1}=0)\,P(X_k=1)$$

$$= q^{k-1}p \quad , \quad k = 1, 2, \ldots$$

These are the geometric(p) probabilities and the corresponding pf (probability function) is

$$f(x) = q^{x-1}p \quad , \ x = 1, 2, \ldots$$
$$= 0 \quad , \ \text{ow}$$

Similarly, for the Poisson(λ) the pf is

$$f(x) = e^{-\lambda}\frac{\lambda^x}{x!} \quad , \ x = 0, 1, \ldots$$
$$= 0 \quad , \ \text{ow}$$

The time to the r th 1 is a negative binomial rv. Call it S_r. Clearly S_r is the sum of r iid geometric(p) rv's. Hence the pgf of S_r is

$$[\text{pgf of a geometric}(p)]^r$$

pgf of a geometric(p)

If $Y \sim$ geometric(p) then

$$G(s) = E(s^Y) = \sum_y s^y f(y) = \sum_{k=1}^{\infty} s^k q^{k-1} p$$

$$= \frac{p}{q} \sum_{k=1}^{\infty} (qs)^k$$

$$= \frac{p}{q} \frac{qs}{1-qs} , \quad \underbrace{|qs|<1}_{|s|<\frac{1}{q}}$$

$$= \frac{ps}{1-qs}$$

Aside

Note $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} , \quad |x|<1$

$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x} , \quad |x|<1$

+ we can differentiate to get related series

If $S_r \sim \underbrace{\text{negative binomial}}_{\text{negbinomial}(r,p)}(r, p)$

$$\Rightarrow G_{S_r}(s) = \left(\frac{ps}{1-qs}\right)^r , \quad |s| < \frac{1}{q}$$

Also $P(S_r = k) = P((r-1)$ 1's in the first $k-1$ trials + 1 on the kth)

$= P((r-1)$ 1's in the first $k-1$ trials$)\, P(1$ on the kth$)$

$$= \binom{k-1}{r-1} p^{r-1} q^{k-r} p = \binom{k-1}{r-1} p^r q^{k-r} , \quad k = r, r+1, \dots$$

The mean & variance of these distributions may be obtained directly or via the pgf. Direct calculations are often harder. An alternative way of obtaining the moments is to use the moment generating function (mgf). For a rv X this is defined as

$$m(t) = E\left(e^{tX}\right) \left\{ = \sum_x e^{tx} f(x) \text{ in the discrete case} \right\}$$

Note: 1. $m(0) = 1$ but $m(t)$ may be ∞ for other t's. The nice case is when $m(t) < \infty$ for $-\epsilon < t < \epsilon$ with $\epsilon > 0$.

2. For counting rv's with pgf $G(s)$

$$\boxed{m(t) = G(e^t)}$$

3. $m^{(k)}(0) = E(X^k)$

eg The pgf of a Poisson(λ) rv is

$$G(s) = e^{\lambda(s-1)}$$

& the mgf is

$$m(t) = e^{\lambda(e^t - 1)}$$

Hypergeometric rv

$$\boxed{\begin{array}{ll} N_1 & b\text{'s} \\ N_2 & w\text{'s} \end{array}}$$

select n chips without replacement

$Y = \#$ of b's

$$P(Y=k) = \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}}$$

where $N = N_1 + N_2$. Had the chips been selected with replacement then $Y \sim \text{Binomial}(n, p)$, where $p = \frac{N_1}{N}$.

The hypergeometric completes our introduction to the main discrete distributions.