The df F

$$F(x) = P(X \le x)$$

right cts at x_0 (for any x_0)

ie $\lim_{x \downarrow x_0} F(x) = F(x_0)$

$\Leftrightarrow$ for any $x_n \downarrow x_0$ $\lim_{n\to\infty} F(x_n) = F(x_0)$

$\rightarrow \lim_{n\to\infty} P(X \le x_n) = P(X \le x_0)$

decreasing sequence of events

want $P\Big(\underbrace{\lim_{n\to\infty} \{X \le x_n\}}_{?}\Big)$

monotone

$$\lim_{x \uparrow x_0} F(x) = \lim_{x_n \uparrow x_0} P(\{X \le x_n\})$$

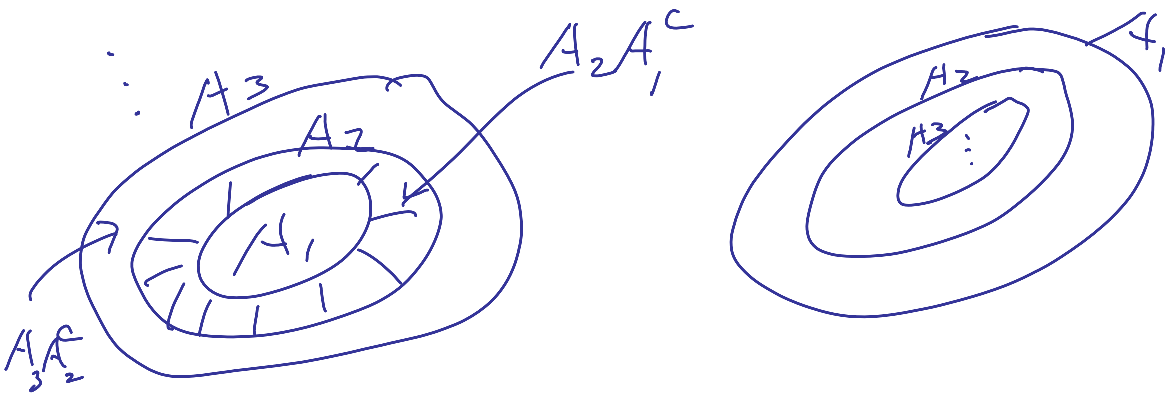
$$= \lim_{n\to\infty} P(\{X \le x_n\})$$

increasing sequence of events

$$\overset{??}{=} P\left(\lim_{n\to\infty} \{X \le x_n\}\right)$$

Events $\quad A_1 \subset A_2 \subset \cdots$ $\quad$ inc

$\quad\quad\quad A_1 \supset A_2 \supset \cdots$ $\quad$ dec



$$\lim_{n\to\infty} P(A_n) \; ?$$

In the inc case $\quad P(A_n) \to P\left(\bigcup_{k=1}^{\infty} A_k\right)$

" " dec " $\quad P(A_n) \to P\left(\bigcap_{k=1}^{\infty} A_k\right)$

Call $\bigcup_{k=1}^{\infty} A_k = \lim_{n\to\infty} A_n$ in the inc case

$\quad\quad \bigcap_{k=1}^{\infty} A_k =$ " $\quad$ " dec "

Continuity Property of P

If $A_n \uparrow\downarrow A \Rightarrow P(A_n) \to P(A)$

In the inc case

$$P(A_n) \to P\left(\bigcup_{k=1}^{\infty} A_k\right)$$

$$A_n = A_1 \cup (A_2 A_1^c) \cup \cdots \cup (A_n A_{n-1}^c)$$

$$\Rightarrow P(A_n) = P(A_1) + P(A_2 A_1^c) + \cdots + P(A_n A_{n-1}^c)$$

$$= \sum_{k=1}^{n} P(A_k A_{k-1}^c) \qquad (A_0 = \emptyset)$$

$$\to \sum_{k=1}^{\infty} P(A_k A_{k-1}^c)$$

$$= P\left(\bigcup_{k=1}^{\infty} A_k A_{k-1}^c\right) = P\left(\overbrace{\bigcup_{k=1}^{\infty} A_k}^{\lim_{n\to\infty} A_n}\right)$$

$$\underbrace{A_n \downarrow A}_{} \Rightarrow \underbrace{A_n^c \uparrow A^c}_{\Downarrow} \left\{= \bigcup_{k=1}^{\infty} A_k^c\right\}$$

$$A = \bigcap_{k=1}^{\infty} A_k$$

$$P(A_n^c) \to P(A^c)$$

$$\Rightarrow 1 - P(A_n) \to 1 - P(A) \Rightarrow P(A_n) \to P(A)$$

Application (to df)

$x_n \downarrow x_0$ $$\lim_{n\to\infty} P(\{X \le x_n\})$$

$$= P\left(\lim_{n\to\infty}\{X \le x_n\}\right)$$

$$= P\Big(\underbrace{\bigcap_{k=1}^{\infty}\{X \le x_k\}}_{\{X \le x_0\}}\Big) = F(x_0)$$

$$\lim_{x_n \uparrow x_0} P(X \le x_n) = P\left(\bigcup_{k=1}^{\infty}\{X \le x_k\}\right)$$

$$= P(\{X < x_0\})$$

Note $$\boxed{P(X = x_0) = F(x_0) - \lim_{x \uparrow x_0} F(x)}$$

df F

- $0 \le F \le 1$, $F(-\infty) = 0$, $F(+\infty) = 1$
- increasing
- rt cts

$\bar{F} = 1 - F$ ← (right) tail probability f'n

rv X $\quad F(x) = P(X \le x)$, $-\infty < x < \infty$

cts df's

Set $f(x) = F'(x) \quad (= -\bar{F}'(x))$

$$\int_{-\infty}^{\infty} f(x)\,dx \overset{FLofC}{=} F(x)\Big|_{-\infty}^{\infty} = \underbrace{F(\infty) - F(-\infty)}_{1}$$

If this happens we have an (absolutely) cts prob dist'n & f is the pdf;

$P(a \underset{(<)}{\le} X \underset{(<)}{\le} b)$

$$\boxed{f(x)\,dx \simeq P(x < X < x + dx)}$$

In the discrete case

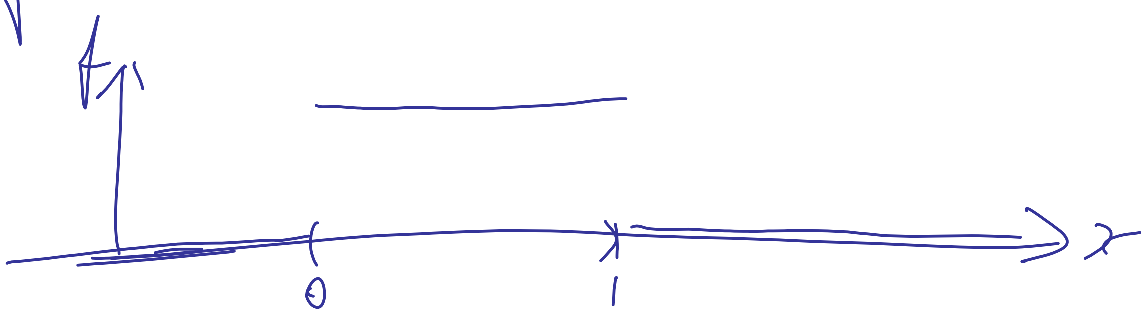
$$E[g(X)] = \sum_{\text{all } x} g(x) \underbrace{f(x)}_{\text{pf}} \quad - \text{"}\forall\text{"} g$$

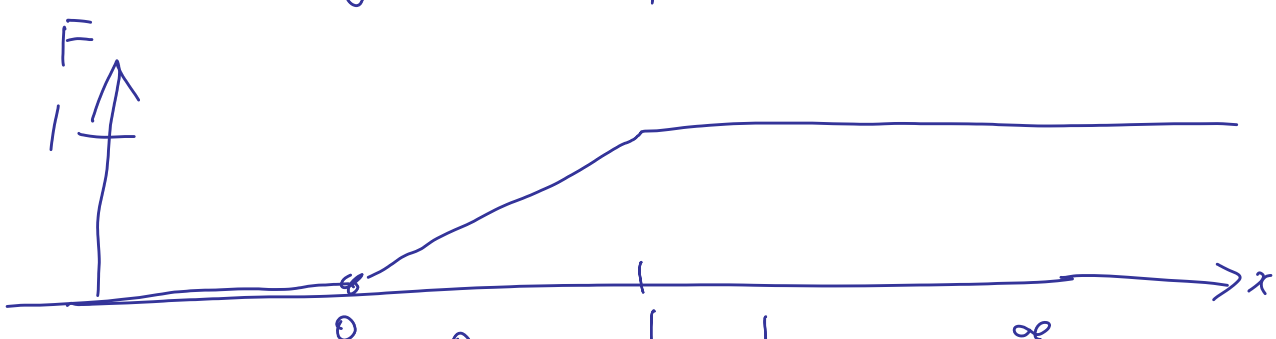
after a few weeks / years / ... we get almost the same result in the cts case

Assume true
$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \quad - \text{"}\forall\text{"} g}$$

uniform $(0,1)$ rv

pdf f is constant on $(0,1)$





$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{0} x \times 0 \, dx + \int_{0}^{1} x \times 1 \, dx + \int_{1}^{\infty} x \times 0 \, dx = \frac{1}{2}$$

$\boxed{\text{From a uniform can get any other rv via a suitable transformation}}$

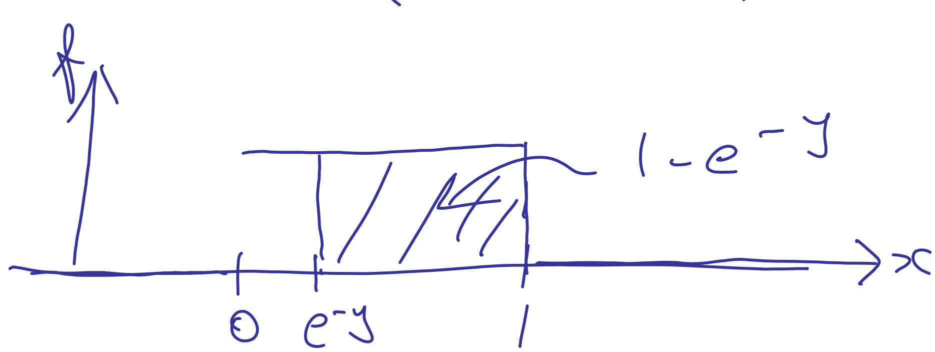
Let $X \sim$ uniform $(0,1)$ rv. Set

$$Y = -\log(X)$$

Note $Y > 0$. What kind of rv is it?

Let $y > 0$. Then

$$F_Y(y) = P(Y \leq y)$$
$$= P(-\log X \leq y) = P(\log X \geq -y)$$
$$= P(e^{\log X} \geq e^{-y})$$
$$= P(X \geq e^{-y}) = 1 - e^{-y}$$



$\therefore$ $F_Y(y) = 1 - e^{-y}$, $y > 0$
$= 0$, ow

$\Rightarrow$ $f_Y(y) = F_Y'(y)$
$= e^{-y}$, $y > 0$
$= 0$, ow

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ exponential(1) pdf

<u>mgf of Y</u>

$$m(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f_Y(y)\,dy$$

$$= \int_0^{\infty} e^{ty} e^{-y}\,dy = \int_0^{\infty} e^{-(1-t)y}\,dy$$

$$= \frac{1}{1-t} \quad , \quad t < 1$$

$$\boxed{m^{(k)}(0) = E(Y^k)}$$

Here, $m^{(1)}(t) = \dfrac{1}{(1-t)^2}$, $t<1$

$m^{(2)}(t) = \dfrac{2}{(1-t)^3}$, $t<1$

$$\left.\begin{aligned} \Rightarrow m^{(1)}(0)=1 &\leftarrow E(Y)=1 \\ m^{(2)}(0)=2 &\leftarrow E(Y^2)=2 \end{aligned}\right\} \operatorname{Var}(Y) = E(Y^2) - [E(Y)]^2 = 1$$

<u>Final note on the df F</u>

Let $x_n \uparrow \infty$. Then $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} P(\{X \le x_n\})$

$$= P(\lim_{n\to\infty} \{X \le x_n\})$$

$$= P(\bigcup_n \{X \le x_n\}) = P(X<\infty) = 1,$$

& hence $\lim_{x\to\infty} F(x)=1$. Write this as $F(\infty)=1$.

If $x_n \downarrow -\infty$ then $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} P(\{X \le x_n\})$

$$= P(\lim_{n\to\infty} \{X \le x_n\})$$

$$= P(\bigcap_n \{X \le x_n\}) = P(\emptyset) = 0,$$

& hence $\lim_{x\to-\infty} F(x) = 0$. Write this as $F(-\infty)=0$.