Week 7

$y = ax + b$; $y = x_1 + x_2 + \cdots + x_n$

X, $P(X \in B)$, $F(x) = P(X \le x)$, $m(t) = E(e^{tX})$

$f(x) = F'(x) = -\bar{F}'(x)$

$F(x) = \int_{-\infty}^{x} f(t)\,dt \leftarrow$ cts

standard exponential

$$f(x) = e^{-x}, \quad x > 0$$
$$= 0, \quad ow$$

$$m(t) = \frac{1}{1-t}, \quad t < 1$$

$$\boxed{m^{(k)}(0) = E(X^k)}$$

For the exponential $m^{(1)}(t) = \frac{1}{(1-t)^2}$, $t < 1$

$m^{(2)}(t) = \frac{2}{(1-t)^3}$, $t < 1$

$t = 0 \longrightarrow E(X) = 1$ & $E(X^2) = 2$ & $Var(X) = 1$

Proposition If $X_1, \dots, X_n$ are independent then $m_{X_1+\cdots+X_n}(t) = m_{X_1}(t) \cdots m_{X_n}(t)$

Proof:
$$m_{X_1+\cdots+X_n}(t) = E\left[e^{t(X_1+\cdots+X_n)}\right]$$
$$= E\left(e^{tX_1} \cdots e^{tX_n}\right)$$
$$= E\left(e^{tX_1}\right) \cdots E\left(e^{tX_n}\right)$$
$$= m_{X_1}(t) \cdots m_{X_n}(t)$$

eg Let $X_1, \dots, X_n$ be iid exponential(1)

Look at
$$Y = X_1 + \cdots + X_n$$
$$m_Y(t) = \left[m_{X_1}(t)\right]^n = \frac{1}{(1-t)^n}, \quad t < 1$$
$$= \int_{-\infty}^{\infty} e^{ty} f_Y(y)\, dy$$

Can't solve it at this point. Could guess.

Back to $Y = aX + b$

Lemma (i) $E(Y) = a\,E(X) + b$

(ii) $\mathrm{Var}(Y) = a^2\,\mathrm{Var}(X)$

Suppose Y has mean μ & standard deviation $\sigma > 0$. Let

$$Z = \frac{Y - \mu}{\sigma} \quad \text{(standardizing a rv)}$$

$$E(Z) = \frac{1}{\sigma} E(Y - \mu) = \frac{1}{\sigma}\left[E(Y) - \mu\right] = 0$$

$$\mathrm{Var}(Z) = \frac{1}{\sigma^2}\mathrm{Var}(Y - \mu) = \frac{\mathrm{Var}(Y)}{\sigma^2} = 1$$

Proposition Let $X_1, \dots, X_n$ be independent. Then

$$\mathrm{Var}(X_1 + \dots + X_n) = \mathrm{Var}(X_1) + \dots + \mathrm{Var}(X_n)$$

<u>Proof</u> Let $\mu_i = E(X_i)$. Then

$$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}\left[(X_1 + \cdots + X_n) - (\mu_1 + \cdots + \mu_n)\right]$$

$$= \mathrm{Var}\left[\underbrace{(X_1 - \mu_1)}_{\text{mean}=0} + \cdots + \underbrace{(X_n - \mu_n)}\right]$$

$$= E\left[\left((X_1 - \mu_1) + \cdots + (X_n - \mu_n)\right)^2\right]$$

$$= E\left[(X_1 - \mu_1)^2 + \cdots + (X_n - \mu_n)^2 + (X_1 - \mu_1)(X_2 - \mu_1) + (X_1 - \mu_1)(X_3 - \mu_1) + \cdots\right]$$

$$= \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) + 0$$

<u>Note</u> – $Y = aX + b$

$$m_Y(t) = E(e^{tY}) = E(e^{atX + bt})$$

$$= e^{bt} E(e^{atX}) = e^{bt} m_X(at)$$

– $m^{(k)}(0) = E(X^k)$

"$\Rightarrow$" $m(t) = \underbrace{m(0)}_{1} + \underbrace{m^{(1)}(0)}_{E(X)}t + \underbrace{m^{(2)}(0)}_{E(X^2)}\frac{t^2}{2} + (\ )$,

where $\boxed{\frac{(\ )}{t^2} \to 0 \text{ as } t \to 0}$.

$(\ )$ is $o(t^2)$

Let $X_1, \dots, X_n$ be iid with mean μ, variance σ^2 + mgf $m_X(t)$.

$$X_1 + \cdots + X_n$$

Aside

Suppose you know the pdf of X, say $f(x)$ + $Y = aX + b$. Then we can
($a > 0$)
easily obtain the pdf of Y. Look at

$$F_Y(y) = P(Y \le y)$$
$$= P(aX + b \le y) = P(aX \le y - b)$$
$$= P\left(X \le \frac{y-b}{a}\right)$$

$$= F_X\left(\frac{y-b}{a}\right)$$

$$\therefore\ f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(if $a<0$ then $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$)

Let $\overline{X} = \dfrac{X_1 + \cdots + X_n}{n}$ ($\approx \mu$ for large n Law of Large #'s)

$$E(\overline{X}) = \frac{1}{n} E(X_1 + \cdots + X_n) = \mu$$

$$Var(\overline{X}) = \frac{1}{n^2} Var(X_1 + \cdots + X_n) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\frac{\bar{X} - \mu}{\underbrace{\sigma/\sqrt{n}}_{SD(\bar{X})}} \quad \mu \leftarrow E(\bar{X})$$

$$= \frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n}\mu\right)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underbrace{\left(\frac{X_i - \mu}{\sigma}\right)}_{Z_i}$$

Note $E(Z_i) = 0$, $Var(Z_i) = 1$. Set $m_Z(t) = E(e^{tZ_i})$.

The mgf of $\frac{1}{\sqrt{n}}(Z_1 + \cdots + Z_n)$

$= \text{mgf of } \frac{Z_1}{\sqrt{n}} + \cdots + \frac{Z_n}{\sqrt{n}}$

$= \left(\text{mgf of } \frac{Z_1}{\sqrt{n}}\right)^n = \left[m_Z\left(\frac{t}{\sqrt{n}}\right)\right]^n$

$$\approx \left(1 + \frac{1\left(\frac{t}{\sqrt{n}}\right)^2}{2}\right)^n$$

$$= \left(1 + \frac{t^2}{2n}\right)^n \to e^{t^2/2}$$

We now have 2 new mgf's

sum of r iid exponential(1)'s $\quad \left(\frac{1}{1-t}\right)^r, \quad t<1$

$$e^{t^2/2}$$

So we want

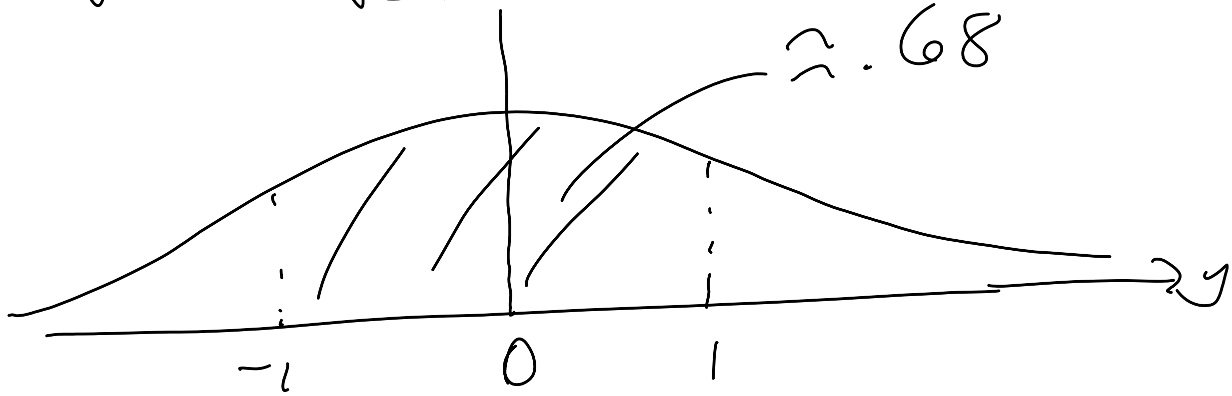
(1) $\displaystyle\int_{-\infty}^{\infty} f(y) e^{ty} dy = \left(\frac{1}{1-t}\right)^r, \quad t<1$

&

(2) $\displaystyle\int_{-\infty}^{\infty} f(y) e^{ty} dy = e^{t^2/2}$

Ex (2)

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad -\infty < y < \infty$$



standard normal — $N(0, 1)$

Note if $Z \sim N(0,1)$ then

$\sigma Z + \mu$ is a $N(\mu, \sigma^2)$ ⇐ def'n

$\sigma > 0$

$$f(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\left(\frac{y-\mu}{\sigma}\right)^2 / 2\right], \quad -\infty < y < \infty$$

For (1) $f(y) = \frac{y^{r-1} e^{-y}}{(r-1)!}$, $y > 0$

$= 0$, ow

gamma(r) pdf

$N(0,1)$

Is $f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

a pdf? ie is the area under $e^{-y^2/2}$ $\sqrt{2\pi}$? Yes!

Aside

Let $g(x) \geq 0$ + $h(y) \geq 0$. Look at

$$g(x)\,h(y) \quad , \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}$$

Fact volume under $g(x) h(y)$ "

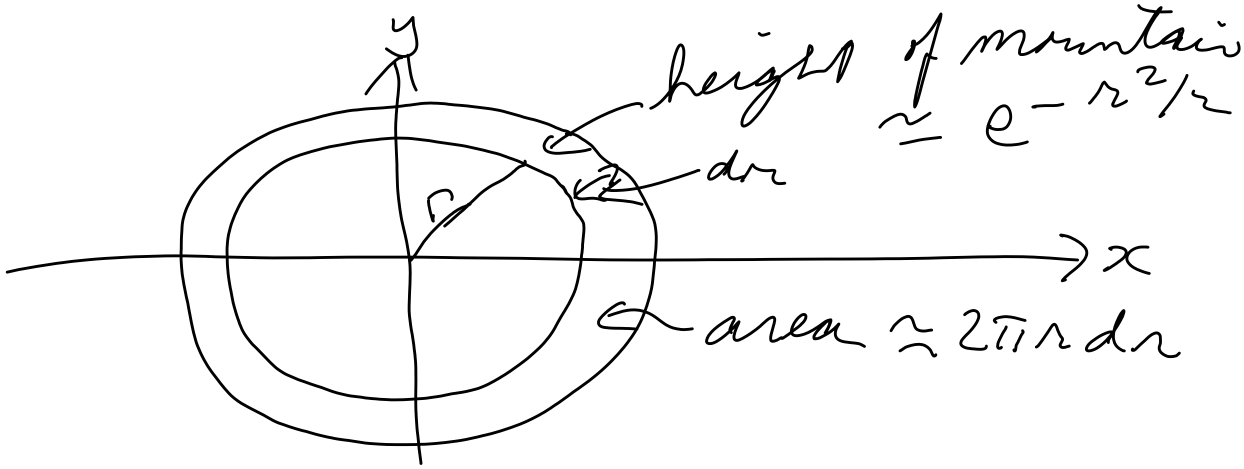
= (area under g) (area under h)

Consider $e^{-x^2/2} e^{-y^2/2}$. The volume under this is 2π.

$\Rightarrow$ area under $e^{-x^2/2}$ is $\sqrt{2\pi}$

$\Rightarrow$ our thing is a pdf

For the volume under $e^{-\frac{x^2+y^2}{2}}$



$\therefore$ $\text{vol} = \int_0^\infty e^{-r^2/2}\, 2\pi r\, dr = 2\pi$

$\therefore$ $f(y) = \dfrac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad -\infty < y < \infty$

is a pdf

<u>Some immediate extensions</u>

<u>exponential (λ)</u> $-$ $\lambda > 0$

If $X \sim$ exponential(1) then $Y = \dfrac{X}{\lambda} \sim$ exponential(λ)

pdf $\quad f(y) = \lambda e^{-\lambda y}, \quad y > 0$

$\qquad\quad = 0 \qquad\quad$, ow

mgf $\quad m(t) = \dfrac{\lambda}{\lambda - t}$, $t < \lambda$ $\Big\}$ using previous results

Note $f_Y(y) = \dfrac{1}{\left(\frac{1}{\lambda}\right)} f_X\left(\dfrac{y}{1/\lambda}\right)$

If $X \sim$ exponential(1) then $E(X) = 1$ & $Var(X) = 1$

If $Y = \dfrac{X}{\lambda} \Rightarrow E(Y) = \dfrac{1}{\lambda}$ & $Var(Y) = \dfrac{1}{\lambda^2}$

λ is often called the rate. Note that the text uses $\frac{1}{\beta}$ for λ.

<u>general normal</u>

If $Z \sim N(0,1)$ then $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$, $\forall z$

& $m_Z(t) = e^{t^2/2}$.

<u>Def'n</u> $Y \sim N(\mu, \sigma^2)$ if $Y = \mu + \sigma Z$, where $Z \sim N(0,1)$.

<u>Note</u> $E(Y) = \mu$, $Var(Y) = \sigma^2$.

$f_Y(y) = \dfrac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\dfrac{1}{2}\left(\dfrac{y-\mu}{\sigma}\right)^2\right]$, $\forall y$; $m_Y(t) = e^{\mu t} e^{\sigma^2 t^2/2}$

gamma(r, λ)

Let $X_1, \cdots, X_r$ be iid exponential(1). Then

$$Y = X_1 + \cdots + X_r \sim \text{gamma}(r)$$

and has pdf + mgf

$$f_Y(y) = \frac{y^{r-1} e^{-y}}{(r-1)!}, \ y>0 \quad ; \quad m_Y(t) = \left(\frac{1}{1-t}\right)^r, \ t<1$$
$$= 0 \quad , \text{ow}$$

Now divide by $\lambda > 0$ so that

$$W = \frac{Y}{\lambda} = \frac{X_1}{\lambda} + \cdots + \frac{X_r}{\lambda} \sim \text{gamma}(r, \lambda)$$

$$f_W(w) = \frac{\lambda^r w^{r-1} e^{-\lambda w}}{(r-1)!}, \ w>0 \quad ; \quad m_W(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, \ t<\lambda$$
$$= 0 \quad , \text{ow}$$

The standard gamma, gamma(r), has pdf

$$f_Y(y) = c\, y^{r-1} e^{-y}, \ y>0$$
$$= 0 \quad , \text{ow}$$

where c is such as to make the area 1.

Set $\Gamma(r) = \int_0^\infty y^{r-1} e^{-y} dy$, $r>0$. Then

$C = \dfrac{1}{\Gamma(r)}$. This defines a pdf for any $r > 0$.

If $W = \dfrac{Y}{r}$ then $W \sim$ gamma(r, λ) is the general gamma & is defined for $r, \lambda > 0$.

Some problems related to this material

1. For the $N(0,1)$ verify

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} e^{tz}\, dz = e^{t^2/2}$$

<u>Hint</u> Look at $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2}\, dz$

2. If $Z \sim N(0,1)$ calculate the df & mgf of $Y = Z^2$.

3. Use the $N(0,1)$ mgf to obtain the $N(\mu,\sigma^2)$ mgf.

4. Verify $\Gamma(r+1) = r\,\Gamma(r)$, $\Gamma(1) = 1$, $\Gamma(n) = (n-1)!$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ <u>Hint:</u> For $\Gamma(\frac{1}{2})$ set $u = \sqrt{y}$ & use the properties of the normal.

5. If $Y \sim N(\mu, \sigma^2)$ then $Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$ & so that $P(a < Y < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$ & hence Y-probabilities can be reduced to Z-probabilities. Use this to calculate $P(\mu-\sigma < Y < \mu+\sigma)$ approximately (use tables, computers, etc ...).

Challenge problems

6. Let X have pdf $f_X(x)$ & suppose $h: \mathbb{R} \to \mathbb{R}$ is strictly increasing and differentiable. Obtain the pdf of $Y = h(X)$ using cdf's.

7. Let X be a cts rv with strictly increasing cdf F. Show (a)

$$Y = F(X) \sim \text{uniform}(0,1)$$

(b) If $U \sim \text{uniform}(0,1)$ then

$$X \stackrel{d}{=} F^{-1}(U),$$

where $\stackrel{d}{=}$ means both sides have the same probability dist'n.