Week 8

$X \overset{d}{=} Y$ means $P(X \in B) = P(Y \in B)$, "$\forall$" B

$\iff E[g(X)] = E[g(Y)]$, "$\forall$" g

Theorem $X \overset{d}{=} Y$ iff $F_X = F_Y$

Theorem $f_X = f_Y \implies X \overset{d}{=} Y$

$G_X = G_Y \implies$ "

$m_X(t) = m_Y(t)$ for t around 0

$\implies X \overset{d}{=} Y$

$$\left.\begin{array}{l} E(\cos tX) = E(\cos tY) \\ E(\sin tX) = E(\sin tY) \end{array}\right\} \forall t \implies X \overset{d}{=} Y$$

Note $e^{itX} = \cos tX + i \sin tX$, where $i = \sqrt{-1}$

$$\underbrace{E(e^{itX})}_{c_X(t)} = E(\cos tX) + i\,E(\sin tX)$$

$c_X(t)$ is the characteristic f'n (cf)

Note $X = Y \Rightarrow X \overset{d}{=} Y$

$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y} \Rightarrow h(\underset{\sim}{X}) \overset{d}{=} h(\underset{\sim}{Y})$

$g \propto h$ means $g(x) = c\, h(x)$, $\forall x$

$\propto$: is proportional to

c: constant

cts rv's

uniform (a, b)

exponential (λ)

$$f(x) \propto e^{-\lambda x}, \quad x > 0$$

gamma (r, λ)

$$f(x) \propto x^{r-1} e^{-\lambda x}, \quad x > 0$$

$Z \sim N(0, 1)$

$$f(z) \propto e^{-z^2/2}, \quad \forall z$$

$Y \overset{d}{=} \mu + \sigma Z \sim N(\mu, \sigma^2)$

$$f(y) \propto e^{-\left(\frac{y-\mu}{\sigma}\right)^2/2}$$

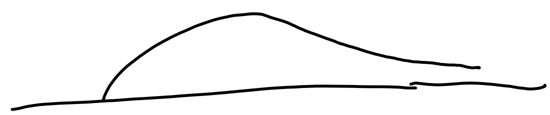
chi-squared dist'n with n degrees of freedom

$Z_1, \dots, Z_n$ iid $N(0,1)$ & let

$$\underset{n\times 1}{\tilde{Z}} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}$$

If $Y \overset{d}{=} |\tilde{Z}|^2 = Z_1^2 + \cdots + Z_n^2$ then

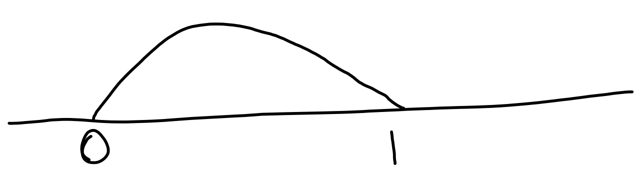
$Y \sim \chi^2(n)$.



Let $$\left.\begin{array}{l} X_1 \sim \text{gamma}(r_1) \\ X_2 \sim \text{gamma}(r_2) \end{array}\right\} \text{ ind}$$

If $Y \overset{d}{=} \dfrac{X_1}{X_1 + X_2}$

then Y is called a beta(r_1, r_2)



Student-t with n degrees of freedom

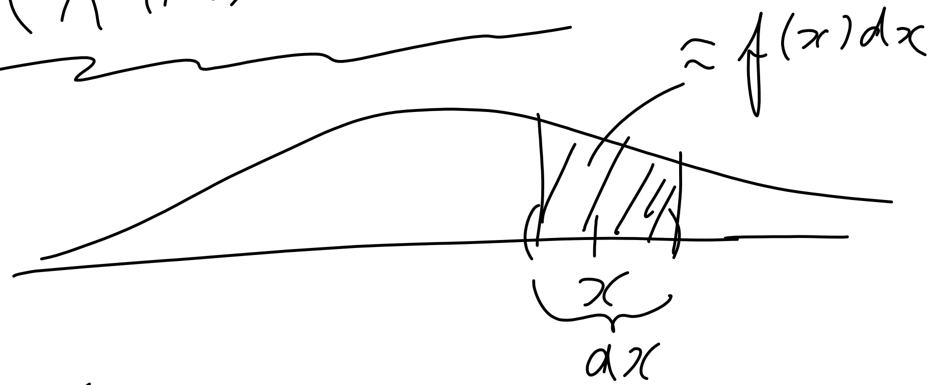
$$Y \overset{d}{=} \frac{Z \sim N(0,1)}{\sqrt{X/n}}, \quad Z, X \text{ ind}, \quad X \sim \chi^2(n)$$

F(m, n) dist'n

$Y \sim F(m,n)$ if

$$Y \overset{d}{=} \frac{(\chi^2(m)\ rv)/m}{(\chi^2(n)\ rv)/n} \quad \text{ind}$$

pdf $f(x)$



$$P(X \in B) = \int_B f(x)\,dx, \quad \text{"}\forall\text{" } B$$

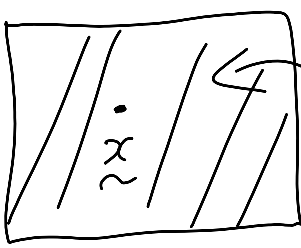
$$E[g(X)] = \int g(x) f(x)\,dx \quad , \text{"}\forall\text{"} g$$

<u>Note</u> $\int g(x) f_1(x)\,dx = \int g(x) f_2(x)\,dx, \ \text{"}\forall\text{"} g$

$\Longrightarrow f_1 = f_2$ <u>almost</u>

The places where they differ forms a set of length 0.

<u>extend to vector case</u> $\underset{m\times 1}{\underset{\sim}{X}} = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}$



volume $= d\underset{\sim}{x} = dx_1 dx_2 \cdots dx_m$

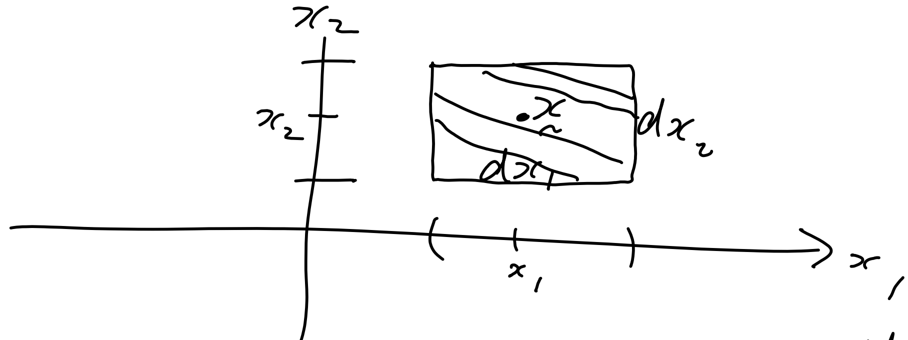
pdf $f(\underset{\sim}{x})$

$P(X \in B) =$ volume over B & under f

$\downarrow$

$\subset \mathbb{R}^m$

<u>IMP</u> $P(\underset{\sim}{X} \in ////) \approx f(\underset{\sim}{x})\, d\underset{\sim}{x}$



If $\underset{\sim}{X}$ has independent components, then

$$P(\underset{\sim}{X} \in \backslash\backslash\backslash) \approx \underbrace{f(\underset{\sim}{x})}_{f(x_1, x_2)}\, d\underset{\sim}{x}$$

$$\|$$

$$P(X_1 \in (\quad),\ X_2 \in |\quad|)$$

$$\|$$

$$P(X_1 \in (\quad))\ P(X_2 \in |\quad|)$$

$$\approx f_1(x_1)\, dx_1\ f_2(x_2)\, dx_2$$

$$\text{"}\Longrightarrow\text{"}\quad f(\underset{\sim}{x}) = f_1(x_1)\, f_2(x_2)$$

In general *if* the components of $\underset{\sim}{X}$ are independent then the pdf of $\underset{\sim}{X}$ is the product of the pdf's of the components.

Application

Let $\underset{\sim}{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ iid $N(0,1)$

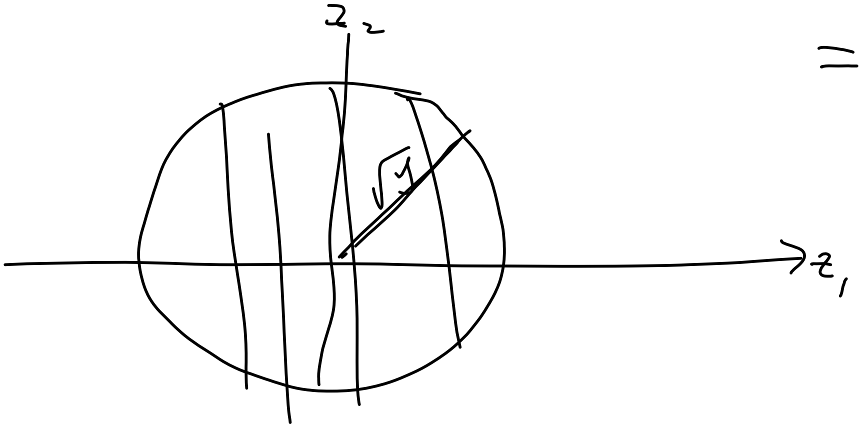
$$\Rightarrow f(\underset{\sim}{z}) = f_1(z_1)\, f_2(z_2)$$
$$= \frac{1}{\sqrt{2\pi}} e^{-z_1^2/2} \frac{1}{\sqrt{2\pi}} e^{-z_2^2/2}$$
$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-|\underset{\sim}{z}|^2/2}$$

Let $Y = Z_1^2 + Z_2^2 \sim \chi^2(2)$

pdf of Y?

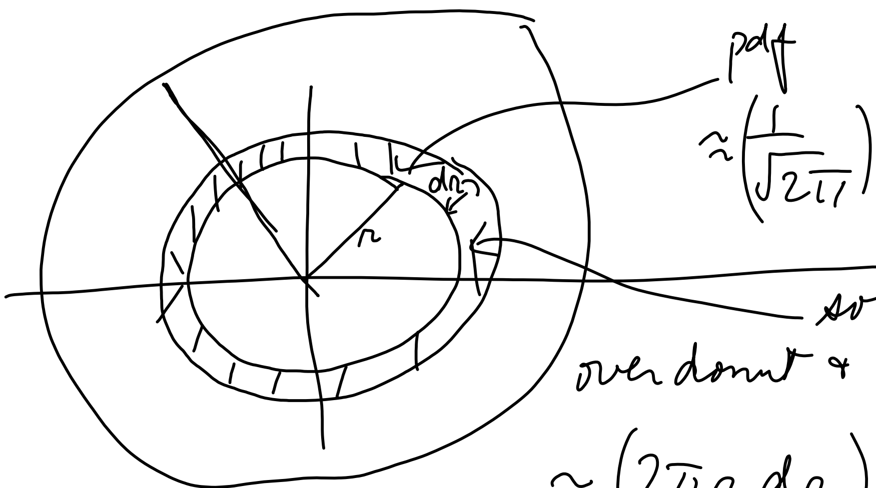
If $y > 0$ then

$$F(y) = P(Y \leq y)$$
$$= P(Z_1^2 + Z_2^2 \leq y) = P(\underset{\sim}{Z} \in \quad)$$



$$= \int f(\underset{\sim}{z})\, d\underset{\sim}{z}$$

$|||$



pdf

$$\approx \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-r^2/2}$$

so volume over donut & under f

$$\approx (2\pi r\, dr)\left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-r^2/2}$$

& so

$$F(y) = \int_0^{\sqrt{y}} 2\pi \left(\frac{1}{\sqrt{2\pi}}\right)^2 r\, e^{-r^2/2}\, dr$$

$$\Rightarrow f(y) = \sqrt{y}\, e^{-y/2} \frac{1}{2\sqrt{y}} = \frac{1}{2} e^{-y/2}$$

If $Y = Z_1^2 + \dots + Z_n^2 \sim \chi^2(n)$

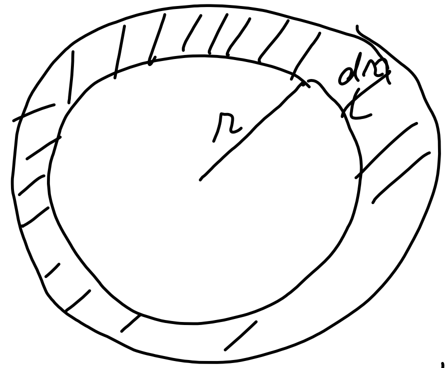
then for $y > 0$

$$F(y) = P(|\underset{\sim}{Z}|^2 \le y)$$

$$= P(|\underset{\sim}{Z}| \le \sqrt{y})$$

volume of a sphere of radius r in n-dim $\propto r^n$

$\Rightarrow$ surface area of a sphere in n-dim $\propto r^{n-1}$



volume of donut $\propto r^{n-1}\,dr$

$\Rightarrow$ volume over donut & under f is

$$\propto r^{n-1}\,dr\,e^{-r^2/2}$$

Note; pdf $f(z) \propto e^{-|\underset{\sim}{z}|^2/2}$

$$\therefore \quad F(y) = c \int_0^{\sqrt{y}} r^{n-1} e^{-r^2/2} \, dr$$

$$\Rightarrow f(y) = c \, y^{\frac{n-1}{2}} e^{-y/2} \frac{1}{2\sqrt{y}}$$

$$= c' \, y^{\frac{n}{2}-1} e^{-y/2}$$

$$\Rightarrow f(y) \propto y^{\frac{n}{2}-1} e^{-y/2}, \quad y > 0$$

which is the pdf of a gamma$\left(\frac{n}{2}, \frac{1}{2}\right)$.

Functions of random vectors (rvec's)

rv X : $G(s) = E(s^X)$, $f(x) = P(X=x)$, $f(x)dx$, $F(x) = P(X \le x)$, $m(t) = E(e^{tX})$, $c(t) = E(e^{itX})$

rvec $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$: $G(\underset{\sim}{s}) = E(s_1^{X_1} \cdots s_n^{X_n})$, $f(\underset{\sim}{x}) = P(\underset{\sim}{X} = \underset{\sim}{x})$,

$f(\underset{\sim}{x})d\underset{\sim}{x}$, $F(\underset{\sim}{x}) = P(\underset{\sim}{X} \le \underset{\sim}{x}) = P(X_1 \le x_1, \ldots, X_n \le x_n)$,

$m(\underset{\sim}{t}) = E(e^{\underset{\sim}{t}'\underset{\sim}{X}}) = E(e^{t_1 X_1 + \cdots + t_n X_n})$,

$c(\underset{\sim}{t}) = E(e^{i\underset{\sim}{t}'\underset{\sim}{X}})$

$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$ if $P(\underset{\sim}{X} \in B) = P(\underset{\sim}{Y} \in B)$, "$\forall$"$B \subset \mathbb{R}^n$

Prop $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y} \Rightarrow g(\underset{\sim}{X}) \overset{d}{=} g(\underset{\sim}{Y})$

Proposition If $X_1 \overset{d}{=} Y_1$, $X_2 \overset{d}{=} Y_2$, $\ldots$, $X_n \overset{d}{=} Y_n$ & $X_1, \ldots, X_n$ are ind as are $Y_1, \ldots, Y_n$ then

$$\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$$

Proof The cf of $\underset{\sim}{X}$ is

$$c_{\underset{\sim}{X}}(\underset{\sim}{t}) = E\left(e^{it_1X_1} \cdots e^{it_nX_n}\right)$$

$$= E\left(e^{it_1X_1}\right) \cdots E\left(e^{it_nX_n}\right), \text{ by ind}$$

$$= E\left(e^{it_1Y_1}\right) \cdots E\left(e^{it_nY_n}\right)$$

$$= E\left(e^{it_1Y_1} \cdots e^{it_nY_n}\right) = c_{\underset{\sim}{Y}}(\underset{\sim}{t})$$

$$\Rightarrow \underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}$$

Recall $Y \sim \text{gamma}(r, \lambda)$ has mgf $m_Y(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^r$, $t < \lambda$

Now let $Y_1, \ldots, Y_n$ be independent gamma's with the <u>same</u> λ. Say $Y_i \sim \text{gamma}(r_i, \lambda)$.

Then $Y_1 + \cdots + Y_n \sim \text{gamma}(r_1 + \cdots + r_n, \lambda)$

The mgf of the sum is the product

$$= \left(\frac{\lambda}{\lambda-t}\right)^{r_1} \cdots \left(\frac{\lambda}{\lambda-t}\right)^{r_n}, \quad t < \lambda$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^{r_1+\cdots+r_n}, \quad t < \lambda$$

$\Rightarrow$ the result.

Also, sums of independent normals are normal. Sums of " Poisson's " Poisson. " " " binomial's (same p) are binomial.

Let $X \sim \chi^2(n)$ be independent of $Y \sim \chi^2(m)$. Then

$$X+Y \sim \chi^2(n+m)$$

Sol'n #1 Use gamma & done ...

Sol'n #2

$$X \overset{d}{=} Z_1^2 + \cdots + Z_n^2 \qquad \left(\text{the } Z\text{'s are iid } N(0,1)\right)$$

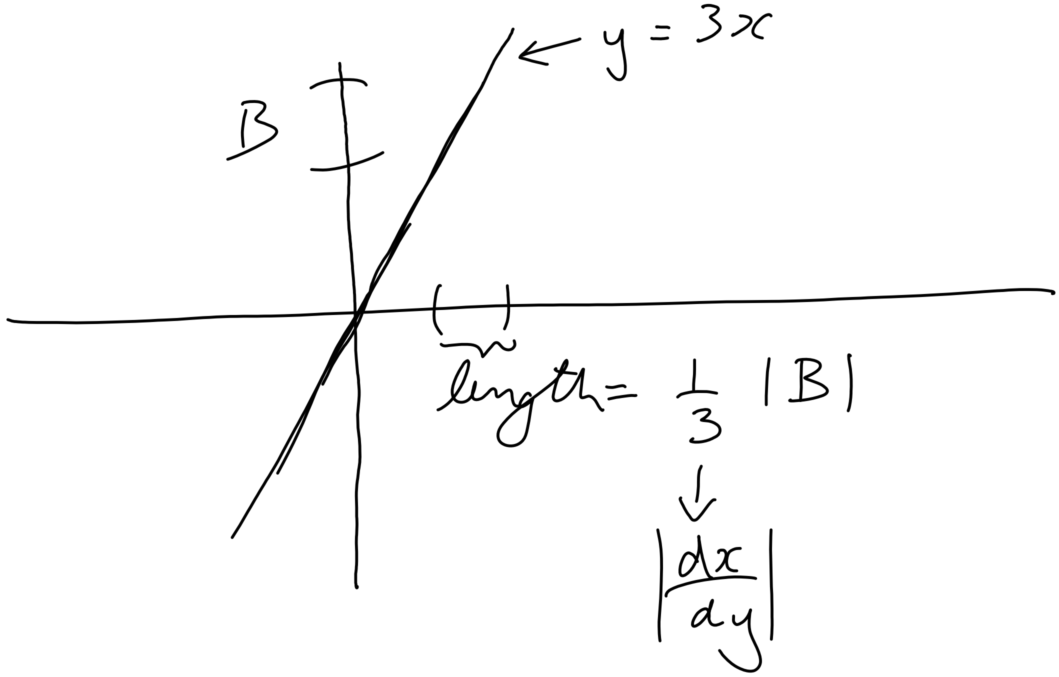
$$Y \overset{d}{=} W_1^2 + \cdots + W_m^2 \qquad \left(\text{the } W\text{'s are iid } N(0,1)\right)$$

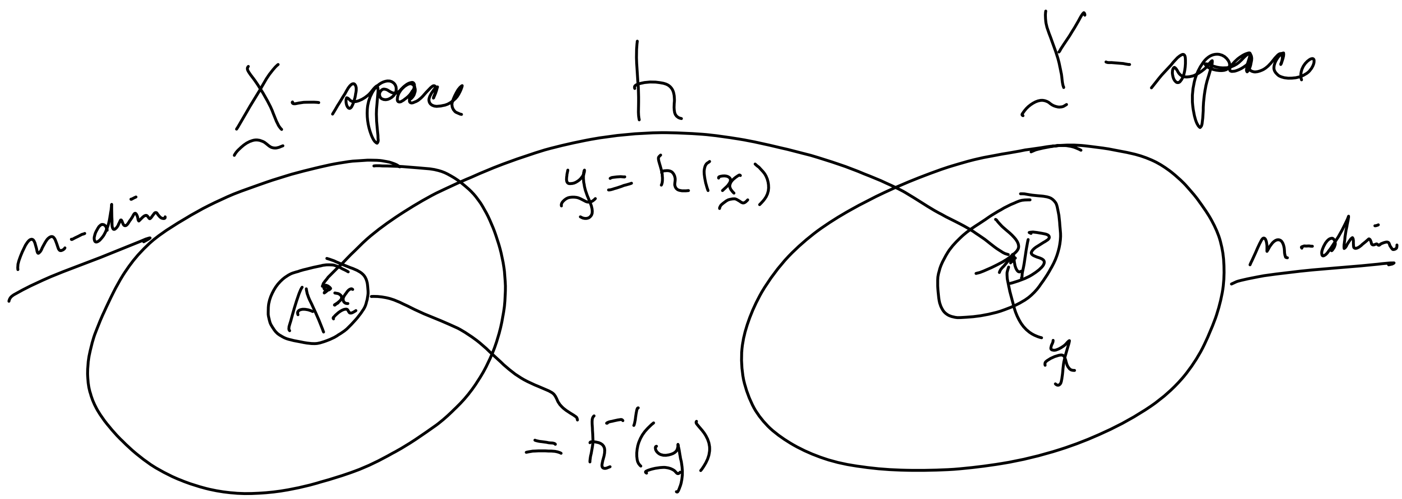
Look at

$$\begin{pmatrix} X \\ Y \end{pmatrix} \overset{d}{=} \begin{pmatrix} Z_1^2 + \cdots + Z_n^2 \\ W_1^2 + \cdots + W_m^2 \end{pmatrix}$$

$$\Rightarrow \quad X + Y \overset{d}{=} \left(Z_1^2 + \cdots + Z_n^2\right) + \left(W_1^2 + \cdots + W_m^2\right)$$

$$\sim \chi^2(n+m)$$





$$|A| \approx \left|\det\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}'}\right)\right| |B| \quad ,$$

where $$\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}'} = \left(\frac{\partial \underset{\sim}{x}}{\partial y_1} \;\; \frac{\partial \underset{\sim}{x}}{\partial y_2} \;\; \cdots \;\; \frac{\partial \underset{\sim}{x}}{\partial y_n} \right)$$

$n \times n$ <u>Jacobian</u>

Now
$$P(\underset{\sim}{X} \in A) = P(\underset{\sim}{Y} \in B)$$
$$\underset{\approx}{} \qquad\qquad \underset{\approx}{}$$
$$f_{\underset{\sim}{X}}(\underset{\sim}{x})\,|A| \qquad f_{\underset{\sim}{Y}}(\underset{\sim}{y})\,|B|$$

" $\Longrightarrow$ " $$\boxed{f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = f_{\underset{\sim}{X}}(h^{-1}(\underset{\sim}{y})) \left|\det\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}'}\right)\right|}$$

change of variables