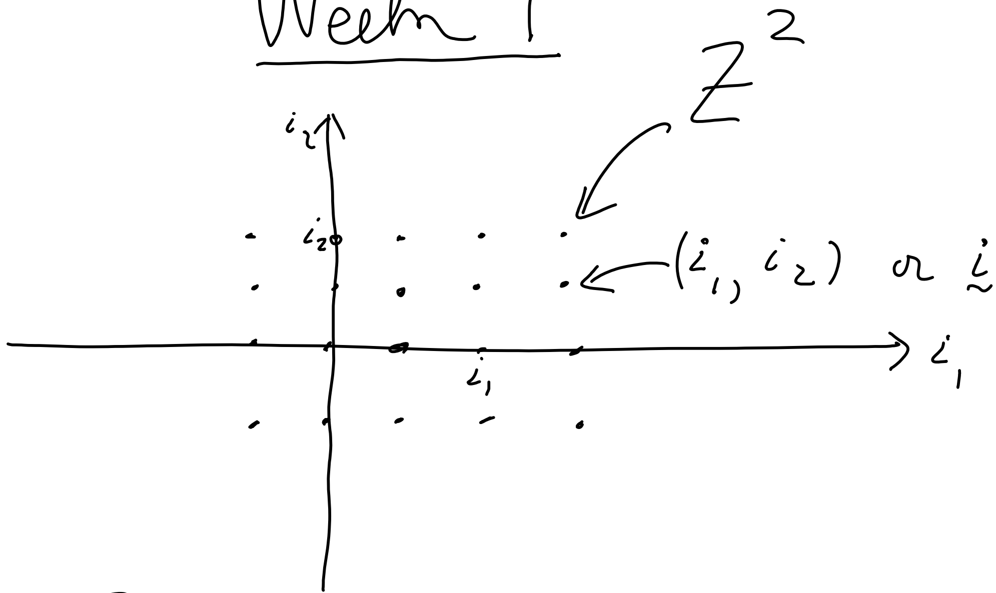
Week 9



$h : Z^2 \longrightarrow \mathbb{R}$

Recall $h = h^+ - h^-$

$$\sum_{\underset{\sim}{i} \in Z^2} h(\underset{\sim}{i}) \quad \text{usually the same} \quad \sum_{i_2} \left[\sum_{i_1} h(i_1, i_2) \right]$$

or

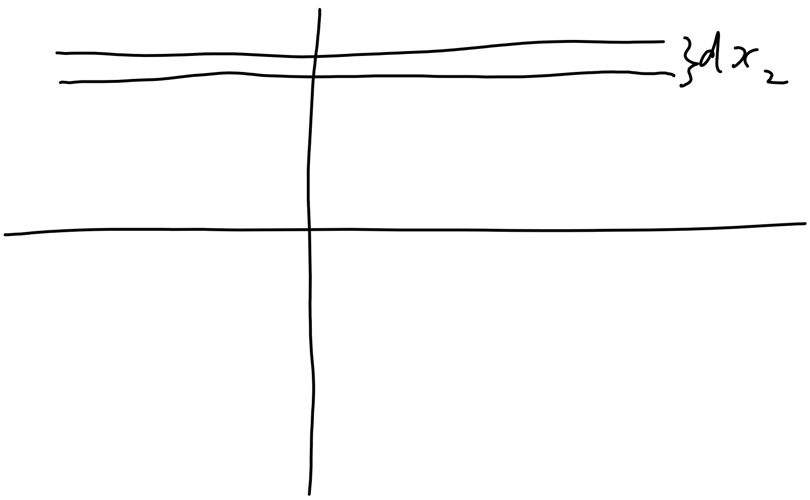
$$\sum_{i_1} \left[\sum_{i_2} h(i_1, i_2) \right]$$

One way

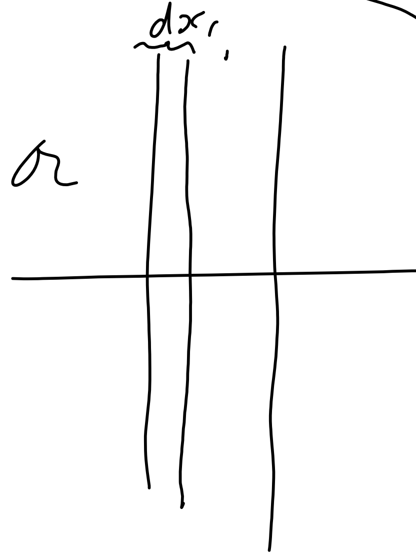
Reduce the problem to two 1-dim sums. If $h : Z^n \to \mathbb{R}$ then we would reduce to n 1-dim sums.

If $h : \mathbb{R}^2 \to \mathbb{R}$ then

$$\int_{\mathbb{R}^2} h(\underset{\sim}{x})\, d\underset{\sim}{x} \quad \left(\int h(\underset{\sim}{x})\, d\underset{\sim}{x} \quad \text{or} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x_1, x_2)\, dx_1\, dx_2 \right)$$



or



$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(x_1, x_2)\, dx_1 \right] dx_2$$

or

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(x_1, x_2)\, dx_2 \right] dx_1$$

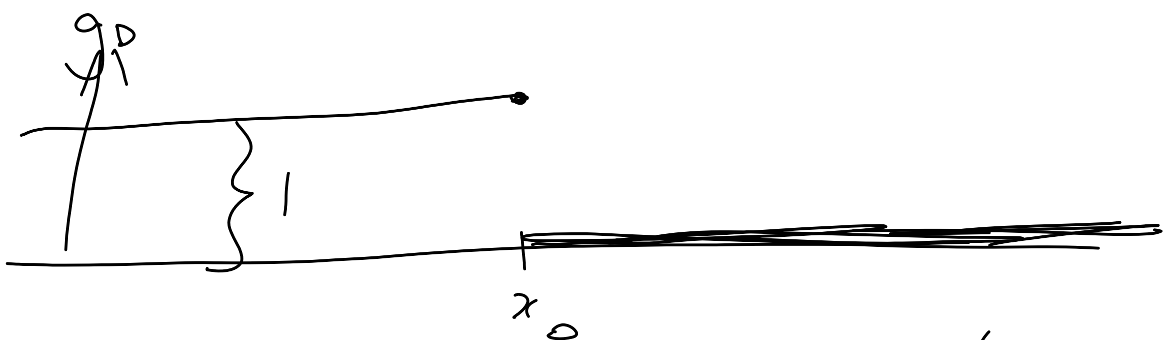
equal and yield

Can extend to $h : \mathbb{R}^n \to \mathbb{R}$.

Back to the course.

X rv, $f(x) = P(X = x)$ or $f(x)dx$

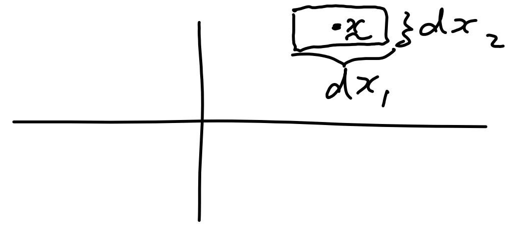
$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$



$$E[g_0(X)] = E(I(\{X \le x_0\})) = P(X \le x_0)$$
$$= F(x_0)$$

For $\underline{\text{rvec}}$'s $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ we

have



$$f(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$\approx P\left(\underset{\sim}{X} \in \quad \right)$$

& $$P\left(\underset{\sim}{X} \in B\right) = \int_B f(\underset{\sim}{x})\, d\underset{\sim}{x}$$

= volume over B & under f

If $B = \mathbb{R}^2$ this is one.

Let $g : \mathbb{R}^2 \to \mathbb{R}$ then

$$E\left[g(\underset{\sim}{X})\right] = \int_{\mathbb{R}^2} g(\underset{\sim}{x})\, f(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(x_1, x_2) f(x_1, x_2) \, dx_1 \right] dx_2$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(x_1, x_2) f(x_1, x_2) \, dx_2 \right] dx_1$$

Now ~~suppose~~

$$f(x_1, x_2) = f(x_1) \, f(x_2)$$

Then

$$E\Big[\underbrace{g_1(X_1) \, g_2(X_2)}_{\text{f'n of } \underset{\sim}{X}}\Big] \overset{?}{=} E\big[g_1(X_1)\big] \, E\big[g_2(X_2)\big], \quad \text{"}\forall\text{"} \; g_1, g_2$$

$$\|$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x_1) \, g_2(x_2) \, f(x_1, x_2) \underbrace{dx_1 dx_2}_{d\underset{\sim}{x}}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g_1(x_1) \, g_2(x_2) \, f(x_1, x_2) \, dx_1 \right] dx_2$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g_1(x_1) g_2(x_2) f(x_1) f(x_2)\, dx_1 \right] dx_2$$

$$= \int_{-\infty}^{\infty} g_2(x_2) f(x_2) \underbrace{\left[\int_{-\infty}^{\infty} g_1(x_1) f(x_1)\, dx_1 \right]}_{E[g_1(X_1)]} dx_2$$

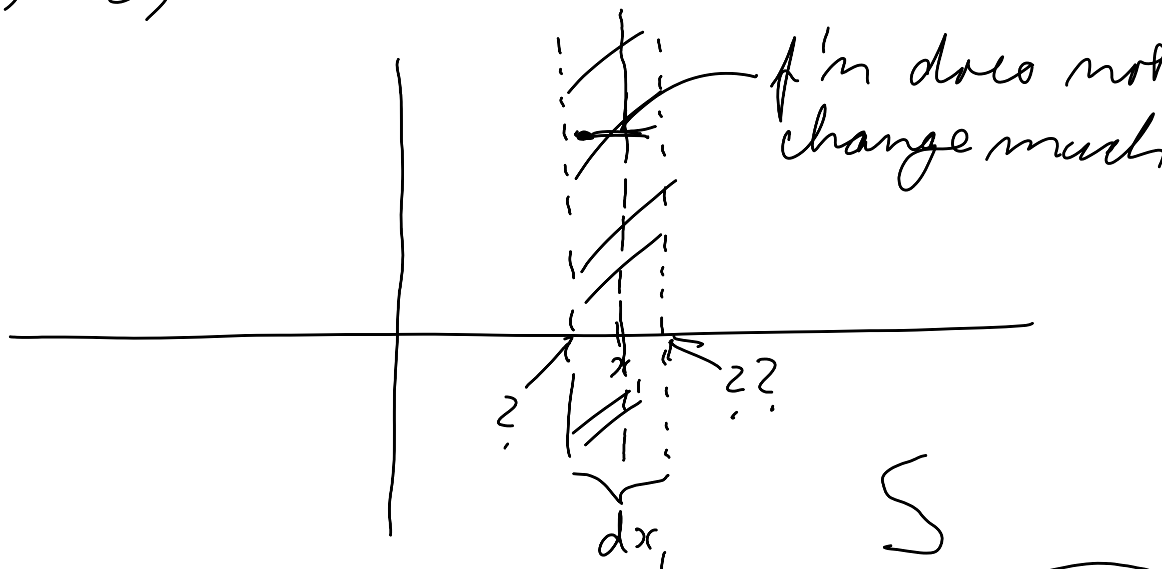
$$= E[g_1(X_1)]\, E[g_2(X_2)] \quad , \text{“}\forall\text{” } g_1, g_2$$

$\Rightarrow X_1$ & X_2 are <u>ind</u>

<u>Note</u> If $f(x_1, x_2) = (f\text{'n of } x_1)(f\text{'n of } x_2)$

$\Rightarrow f(x_1, x_2) = f(x_1)\, f(x_2)$ } problem

$f(x_1, x_2)$



$$P(\underset{\sim}{X} \in ///) = P(? < X_1 < ??,\ \overbrace{-\infty < X_2 < \infty}^{S})$$

$$= P(? < X_1 < ??)$$

$$\approx f(x_1)\,dx_1$$

$\int_{///} f(x_1, x_2)\,d\underset{\sim}{x}$

$$\approx \left[\int_{-\infty}^{\infty} f(x_1, x_2)\,dx_2\right] dx_1 \quad \therefore \text{ "} \therefore \text{ "}$$

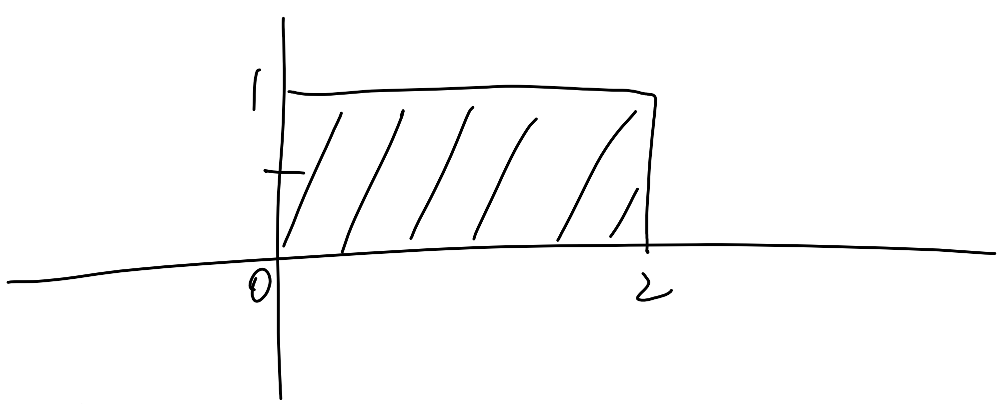
$$f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\,dx_2$$

Also $f(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$

Note $f(\underset{\sim}{y}) = \int_{\mathbb{R}^n} f(\underset{n\times 1}{\underset{\sim}{x}}, \underset{m\times 1}{\underset{\sim}{y}})\, d\underset{\sim}{x}$

eg. Let

$$f(x, y) = c\, x\, y \quad , \quad \underset{\sim}{x} \in ////$$
$$= 0 \quad , \quad \text{OW}$$



c? $\int_{\mathbb{R}^2} f(\underset{\sim}{x})\, dx = 1$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y)\, dx \right] dy$$

$$= \underbrace{\int_{-\infty}^{0} [\quad] dy}_{0} + \underbrace{\int_{1}^{\infty} [\quad] dy}_{0}$$

$$+ \int_{0}^{1} [\quad] dy$$

$$= \int_{0}^{1} \left[\int_{-\infty}^{\infty} f(x,y) dx \right] dy$$

$$= \int_{0}^{1} \left[\int_{0}^{2} cxy\, dx \right] dy$$

$$= c \int_{0}^{1} y \int_{0}^{2} x\, dx\, dy = c \int_{0}^{1} y\, dy \int_{0}^{2} x\, dx$$

$$= c \ \frac{1}{2} x^2 = c$$

$$\Rightarrow \quad \underline{c = 1}$$

eg Let $f(x,y) = e^{-x}e^{-y}, \quad x,y>0$

$= 0, \quad ow$

Are X & Y ind? Yes

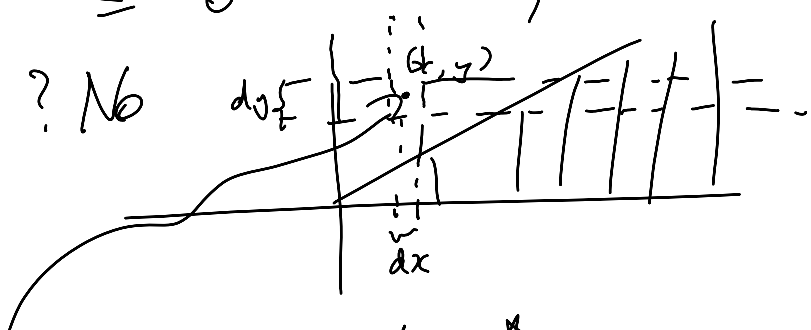
$\therefore$ $f(x,y) = f(x)f(y)$ where $f(x) = e^{-x},\ x>0$ & 0, ow

$f(y) = e^{-y},\ y>0$ & 0, ow

eg Let $f(x,y) = ce^{-x}e^{-y}, \quad 0<y<x<\infty$

$= 0, \quad ow$

Are X & Y ind? No



$f(x,y) = 0$ for this (x,y) but

$f(x) > 0$ & $f(y) > 0$

$$\boxed{f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy > 0}$$

$$\boxed{f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx} > 0$$

Recall

$X_1 \sim gamma(r_1)$ ← ind

$X_2 \sim gamma(r_2)$ ←

pdf for X_i is

$$f(x_i) = \frac{x_i^{r_i - 1} e^{-x_i}}{\Gamma(r_i)}, \quad x_i > 0$$

The mgf for X_i is

$$m_i(t) = \left(\frac{1}{1-t}\right)^{r_i}, \quad t < 1$$

The mgf of $Y = X_1 + X_2$ is

$$m_Y(t) = \left(\frac{1}{1-t}\right)^{r_1 + r_2}, \quad t < 1$$

$\Rightarrow Y \sim gamma(r_1 + r_2)$

$$\Rightarrow f(y) = \frac{y^{r_1 + r_2 - 1} e^{-y}}{\Gamma(r_1 + r_2)}, \quad y > 0$$

Look at

$$Y_1 = \frac{X_1}{X_1 + X_2} \quad \text{is a beta rv}$$

$$Y_2 = X_1 + X_2 \qquad \Big(\underbrace{Y_2 = X_1}_{\text{easier}}\Big)$$

Now we have a 1-1 f'n from $\underset{\sim}{X}$ to $\underset{\sim}{Y}$

$$\Rightarrow f_{\underset{\sim}{Y}}(y) = f_{\underset{\sim}{X}}(\underset{\sim}{x} \text{ in terms of } y) \left| \det\left(\frac{d\underset{\sim}{x}}{d\underset{\sim}{y}'}\right) \right|$$

This yields

$$f(y_1, y_2)$$

$$\Rightarrow f(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2$$

& done.

$$y_1 = \frac{x_1}{x_1+x_2}$$

$$y_2 = x_1 + x_2$$

$$\Rightarrow \quad x_1 = y_1 y_2$$

$$x_2 = y_2 - y_1 y_2$$

$$\frac{d\underset{\sim}{x}}{d\underset{\sim}{y}'} = \begin{pmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{pmatrix}$$

$$\Rightarrow \det\left(\frac{d\underset{\sim}{x}}{d\underset{\sim}{y}'}\right) = y_2$$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

$$= \frac{x_1^{r_1-1} e^{-x_1}}{\Gamma(r_1)} \frac{x_2^{r_2-1} e^{-x_2}}{\Gamma(r_2)}$$

$$f_{\underset{\sim}{X}}(y_1 y_2, y_2 - y_1 y_2) = \frac{(y_1 y_2)^{r_1-1} e^{-y_1 y_2}\, y_2^{r_2-1}(1-y_1)^{r_2-1} e^{-y_2+y_1 y_2}}{\Gamma(r_1)\Gamma(r_2)}$$

$$= y_1^{r_1-1}(1-y_1)^{r_2-1}\; y_2^{r_1+r_2-2} e^{-y_2} / \Gamma(r_1)\Gamma(r_2)$$

$$\Rightarrow f(y_1, y_2) = \frac{y_1^{r_1-1}(1-y_1)^{r_2-1}\,\Gamma(r_1+r_2)}{\Gamma(r_1)\;\Gamma(r_2)} \quad \frac{y_2^{r_1+r_2-1} e^{-y_2}}{\Gamma(r_1+r_2)}$$

$$\Rightarrow f(y_1) = \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\,\Gamma(r_2)}\; y_1^{r_1-1}(1-y_1)^{r_2-1} \;, \quad 0 < y_1 < 1$$

<u>Note</u> - $r_1 = r_2 = 1 \longrightarrow$ uniform $(0, 1)$

- $\dfrac{X_1}{X_1+X_2}$ & $X_1 + X_2$ are independent

Set $\quad Y = \dfrac{X_1}{X_1+X_2}$

$$\Longrightarrow Y(X_1+X_2) = X_1$$

$$\Longrightarrow E(Y)\,E(X_1+X_2) = E(X_1) \quad , \text{by ind}$$

$$\Longrightarrow E(Y) = \frac{E(X_1)}{E(X_1+X_2)} = \frac{r_1}{r_1+r_2}$$

Some little facts

X, pdf $f(x)$; $Y = h(X)$; pdf of Y?

$Y = X + b$ pdf ✓

$Y = cX$ pdf ✓, $c > 0$

$Y = \frac{1}{X}$ pdf ✓

$Y = \frac{1}{1+X}$ pdf ✓

$X_1 \sim \chi^2(m_1)$

$X_2 \sim \chi^2(m_2)$

ind

$$Y = \frac{X_1/m_1}{X_2/m_2} \sim F(m_1, m_2)$$

pdf of Y?

$$Y = c\,\frac{X_1}{X_2}$$

$\Rightarrow$ know if we can get the pdf of

$$W = \frac{X_1}{X_2}$$

Now

$$\frac{X_1}{X_2} + 1 = \frac{X_1 + X_2}{X_2}$$

& so <u>if</u> we can get the pdf of

$$\frac{X_1 + X_2}{X_2}$$

Then we can get the pdf $\frac{X_1}{X_2}$ & then of Y. But

$$\frac{X_1 + X_2}{X_2} = \frac{1}{\frac{X_2}{X_1 + X_2}} \leftarrow \text{beta}$$

& we are done!

eg Z_1, Z_2 iid $N(0,1)$

$$Y_1 = \frac{Z_1}{Z_2} \qquad Y_2 = Z_2$$

$$\left.\begin{array}{l} y_1 = \dfrac{z_1}{z_2} \\ y_2 = z_2 \end{array}\right\} \Rightarrow \begin{array}{l} z_1 = y_1 y_2 \\ z_2 = y_2 \end{array}$$

$$\Downarrow$$

$$\frac{d\underset{\sim}{z}}{d\underset{\sim}{y}'} = \begin{pmatrix} y_2 & y_1 \\ 0 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\det(\ ) = y_2$$

+ so

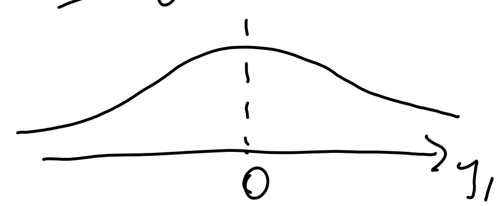
$$f(z_1, z_2) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-(z_1^2 + z_2^2)/2}$$

$$f_{\underset{\sim}{Y}}(y_1, y_2) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-(y_1^2 y_2^2 + y_2^2)/2} \, |y_2|$$

$$\Rightarrow f(y_1) = \frac{1}{\pi} \int_0^\infty e^{-y_2^2(1+y_1^2)/2} \, y_2 \, dy_2$$

$$= \frac{1}{\pi(1+y_1^2)}$$

long tails



Cauchy

If $Y \sim$ Cauchy then

$$E(|Y|) = \int_{-\infty}^{\infty} |y| \frac{1}{\pi(1+y^2)}\, dy = \infty$$

Cauchy has no moments.

$$E\left(e^{\overset{\sqrt{-1}}{i}tY}\right) = e^{-|t|}$$

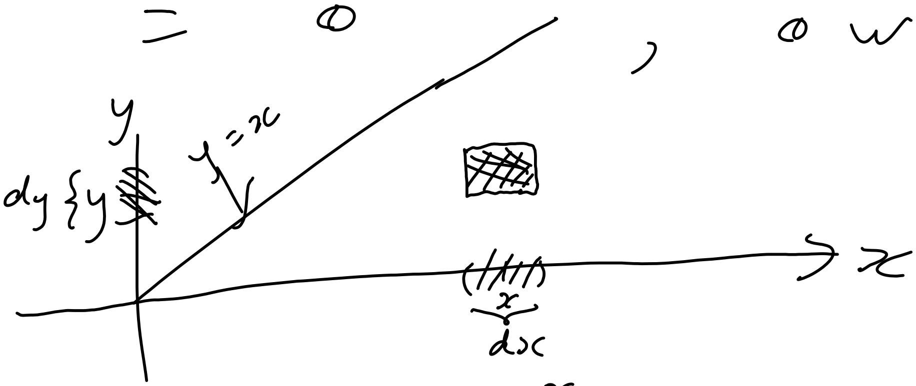
$$= \int_{-\infty}^{\infty} (\cos ty) \frac{1}{\pi(1+y^2)}\, dy$$

If $Y_1, \dots, Y_n$ are iid Cauchy then

$$\bar{Y} = \frac{Y_1 + \dots + Y_n}{n} \quad \& \quad E\left(e^{it\bar{Y}}\right) = e^{-|t|}$$

eg $f(x,y) = c\, e^{-x} e^{-y}$, $0 < y < x < \infty$

$ = 0$, ow



if $x > 0$,

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x c e^{-x} e^{-y}\,dy$$

$$= c\, e^{-x}(1 - e^{-x})$$

BTW $c = 2$

$$P(X \in ///, Y \in \backslash\backslash\backslash) = P(\underset{\sim}{X} \in \#)$$

$$\approx f(x,y)\,dx\,dy$$

& so

$$P(Y \in \backslash\backslash\backslash \mid X \in ////) \approx \frac{f(x,y)\,dx\,dy}{f(x)\,dx}$$

$\frac{f(x,y)}{f(x)}$ is a pdf (wrt y)

This is the conditional pdf of Y given $X = x$.

$$f(y|x)$$

The mean of this is

$$\underbrace{E(Y|X=x)}_{r(x)} = \int_{-\infty}^{\infty} y\, f(y|x)\, dy$$

(regression of Y on X)

$r(X)$ is denoted by $\overbrace{E(Y|X)}^{rv}$

+ it minimizes

$$E\left(\left[Y - f'n(X)\right]^2\right)$$

In the discrete case

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{P(X=x, Y=y)}{P(X=x)}$$

$$= P(Y=y \mid X=x)$$

is the conditional pf.

$$\boxed{f(y|x) = \frac{f(x,y)}{f(x)}}$$

$$\Longrightarrow \boxed{f(x,y) = f(x)\, f(y|x)}$$

ind

eg $U \sim \text{Poisson}(\lambda_1)$, $V \sim \text{Poisson}(\lambda_2)$

$$X = U + V$$

Suppose you know $X = n$. What is the dist'n of V?

$$P(V=k \mid X=m)$$

$$= \frac{P(V=k,\, X=m)}{P(X=m)} = \frac{P(U=m-k,\, V=k)}{P(X=m)}$$

$$= \frac{P(U=m-k)\, P(V=k)}{P(X=m)}$$

$\{X \sim \text{Poisson}(\lambda_1+\lambda_2)\}$

$$\frac{\dfrac{e^{-\lambda_1}\lambda_1^{m-k}}{(m-k)!}\;\dfrac{e^{-\lambda_2}\lambda_2^{k}}{k!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^{m}}{m!}}$$

binomial(m,p)

$p = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$

$$= \binom{m}{k}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{m-k}$$

eg Z_1, Z_2 iid $N(0,1)$

$$Y = \frac{Z_1}{Z_2}$$

Condition on $Z_2 = z_2$

$$\Rightarrow Y = \left(\frac{1}{z_2}\right) Z_1$$

$\Rightarrow$ know $f(y \mid z_2)$

know

$$\Rightarrow f(z_2, y) = f(z_2) f(y \mid z_2)$$

$$\Rightarrow f(y) = \int_{-\infty}^{\infty} f(z_2) f(y \mid z_2) \, dz_2$$

$$= \frac{1}{\pi(1+y^2)}$$

<u>Def'n</u> $\mathrm{cov}(X, Y)$

$$= E(XY) - E(X)\,E(Y)$$

$$= E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

<u>eg</u> $Z \sim N(0, 1)$ let $X = Z$
$Y = Z^2$

$$\mathrm{cov}(X, Y) = \underbrace{E(Z^3)}_{0} - \underbrace{E(Z)}_{0} E(Z^2)$$

$$= 0$$

$\Rightarrow$ X & Y are <u>uncorrelated</u>

But X & Y are dependent.

<u>Note</u> If $\mathrm{cov}(X, Y) \neq 0$ then the variables are correlated. $\rho = \mathrm{corr}(X, Y) = \dfrac{\mathrm{cov}(X, Y)}{SD(X)\,SD(Y)}$ is the correlation (coefficient) between X & Y.