More Problems #1

1.

Let $X_1, X_2$ be iid $X$ where $0 < E(X^2) < \infty$. Suppose

$$\frac{X_1 + X_2}{\sqrt{2}} \stackrel{d}{=} X .$$

Show $X \sim N(0, \sigma^2)$.

2.

(i) Let $a_1$, $a_2$, ... be a sequence. We will denote it by either $\{a_n\}$ or just $a_n$ . Define $a_n \to a$ , as $n \to \infty$ . When there is no confusion we will simply write $a_n \to a$ (or say that $a_n$ converges to a) .

**(ii)If** $a_n$ is a sequence and $n_1 < n_2 < \ldots$then $a_{n_k}$ is called a subsequence of $a_n$ . Show $a_n \to a \Leftrightarrow$ every subsequece of $a_n$ converges to a .

(ii) Suppose every subsequnce of a sequence $a_n$ has a further subsequence which converges to a . Show $a_n \to a$ .

3. It can be shown that $X_n \xrightarrow{p} X$ implies there exists a subsequence $X_{n_k}$ which convereges almost surely to $X$ . Use this fact to prove what one might term a Probabilistic Dominated Convergence Theorem:

Suppose $X_n \xrightarrow{p} X$ and $|X_n| \leq W$ with $E(W) < \infty$ . Show $E(X_n) \to E(X)$ .

4.

Let $X_1, X_2, \cdots$ be iid, $\geq 0$ with continuous df $F$ which is strictly increasing in $x \geq 0$. We say that a <u>record</u> occurs at time $n$ if $X_n > \max\{X_1, \cdots X_{n-1}\}$, $n = 2, 3, \cdots$. Time $n=1$ will by convention be called the initial record time and $X_1$ the initial record value.

(a) Let $T = \min\{n : n > 1$ and $n$ is a record time$\}$

Calculate $P(T > t)$, $P(T < \infty)$ and $E(T)$

(b) Let $T_y = \min\{n : X_n > y\}$. Show that $T_y$ is independent of $X_{T_y}$

(c) Calculate $E[N(t)]$ and $Var[N(t)]$ where $N(t) = \#$ of records up to time $t$

5.

Let $A_1$, $A_2$ , ...be a countably infinite # of events and set

$$Y= \sum_i I_{A_i} \text{ . Show } \{Y=\infty\} = \lim_{n\to\infty} \bigcup_{i=n}^{\infty} A_i \text{ .}$$

6. For the situation in #5 suppose the sum of the probabilities of the A's is finite. Show P(Y=oo) = 0 . On the other hand, if the A's are independent and the sum is oo show P(Y=oo) =1 . These two results form the Borel Cantelli Lemma.

7.

$$\text{Suppose } X_n \xrightarrow{ms} X \text{ . Show } X_n \xrightarrow{p} X \text{ .}$$

8.

$$\text{Let } X_1, X_2, \ldots \text{ be iid uniform}(0,1) \text{ . Show } n(1-X_{(n)}) \xrightarrow{d} \text{exponential}(1) \text{ .}$$

9. A positive rv X is ageless if P(X>s+t|X>s)=P(X>t) , for all s,t>=0 . If X is ageless, and not a constant, show it must be exponential(l) for some l>0.

<u>Remark</u>: In 9 you may not assume X to be a cts rv with some pdf. If F is the df then you must show 1-F(x)=exp(-lx) for x>0 . Since 1-F is right continuous this will be the case if it's true for rational x's .

Solution to #1

3. $\frac{X_1+X_2}{\sqrt{2}} \overset{d}{=} X \Rightarrow \frac{2E(X)}{\sqrt{2}} = E(X) \Rightarrow E(X) = 0$

If $c(t) = E(e^{itX})$, where $i = \sqrt{-1}$, then

$$E(e^{itX}) = E(e^{i\frac{t}{\sqrt{2}}X_1})\,E(e^{i\frac{t}{\sqrt{2}}X_2})$$

$$\Rightarrow \quad c(t) = \left[c\left(\frac{t}{\sqrt{2}}\right)\right]^2$$

$$\Rightarrow \quad c(t) = \left[\left(c\left(\frac{\frac{t}{\sqrt{2}}}{\sqrt{2}}\right)\right)^2\right]^2 = \left[c\left(\frac{t}{(\sqrt{2})^2}\right)\right]^{2^2}$$

$$\vdots$$

$$\Rightarrow c(t) = \left[c\left(\frac{t}{(\sqrt{2})^n}\right)\right]^{2^n} = \left[c\left(\frac{t}{2^{n/2}}\right)\right]^{2^n}$$

$$= \left(1 - \frac{\sigma^2}{2!}\frac{t^2}{2^n} + o\left(\frac{t^2}{2^n}\right)\right)^{2^n}$$

since $c'(0) = i\,E(X) = 0$ & $c''(0) = -E(X^2) = -\sigma^2$.

Now let $n \to \infty$ to get $c(t) = e^{-\sigma^2 t^2/2}$

which is the cf of a $N(0, \sigma^2)$.

<u>Note</u>: $\left[1 + \frac{x}{N} + o\left(\frac{1}{N}\right)\right]^N \to e^x$ as $N \to \infty$

Solution to #4

(a) We have

$$T > n \iff X_1 = \max\{X_1, \cdots, X_n\}$$

so that $P(T > n) = \frac{1}{n}$. Since

$$P(T = \infty) = \lim_{n\to\infty} P(T > n) = 0 \text{ we get } P(T < \infty) = 1.$$

Finally, $E(T) = \sum_{n=1}^{\infty} P(T > n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

(b) Let $T_y$ = time of the first record value $> y$ and set $XT_y$ as the record value at time $T_y$. Then

$$P(XT_y > x \mid T_y = n) = P(X_n > x \mid X_1 < y, \cdots, X_{n-1} < y, X_n > y)$$

$$= P(X_n > x \mid X_n > y)$$

$$= \begin{cases} 1 & \text{if } x < y \\ \dfrac{\bar{F}(x)}{\bar{F}(y)} & \text{if } x > y \end{cases}$$

Since $P(XT_y > x \mid T_y = n)$ does not depend on $n$ $T_y$ is independent of $XT_y$.

(c) $\{\text{record at time } n\} = \{X_n \text{ is the largest of } X_1, \cdots, X_n\}$ and so $P(\{\text{record at time } n\}) = 1/n$. Now

$$N(t) = \sum_{j=1}^{t} I_{\{\text{record at time } j\}}$$

so that $E(N(t)) = \sum_{j=1}^{t} \frac{1}{j}$ + $Var(N(t)) = \sum_{j=1}^{t} \frac{1}{j}\left(1 - \frac{1}{j}\right)$

Solution to #5

$Y = \infty \iff$ an $\infty$ # of $A_1, A_2, \cdots$ occur

$\Rightarrow \bigcup_{i=n}^{\infty} A_i$ occurs for each $n$

$\Rightarrow \underbrace{\bigcap_{n=1}^{\infty} \left( \bigcup_{i=n}^{\infty} A_i \right)}_{\lim_{n \to \infty} \bigcup_{i=n}^{\infty} A_i}$ occurs

Now assume $\lim_{n \to \infty} \bigcup_{i=n}^{\infty} A_i$ occurs

$\Rightarrow \bigcap_{n=1}^{\infty} \left( \bigcup_{i=n}^{\infty} A_i \right)$ occurs

$\Rightarrow \bigcup_{i=n}^{\infty} A_i$ occurs $\forall\, n$

$\Rightarrow$ an $\infty$ # of $A_1, A_2, \cdots$ occur $\Rightarrow Y = \infty$

$\therefore \quad \{ Y = \infty \} = \lim_{n \to \infty} \bigcup_{i=n}^{\infty} A_i$

Note $Y = \infty$ is short for $\{ Y = \infty \}$ occurs

Solution to #7

$$X_n \xrightarrow{ms} X \Rightarrow E(X_n - X)^2 \to 0$$

$$\Rightarrow P(|X_n - X| \geq \epsilon) \leq \frac{E(X_n - X)^2}{\epsilon^2} \to 0$$

$$\Rightarrow X_n \xrightarrow{P} X$$

Solution to #8

(c) Set $Y_n = n(1-X_{(n)})$. Then for $y>0$,

$$P(Y_n>y) = P(n(1-X_{(n)})>y)$$
$$= P(X_{(n)}<1-\frac{y}{n})$$
$$= P(X_i<1-\frac{y}{n}\ ;\ i=1,\cdots,n)$$
$$= (1-\frac{y}{n})^n \to e^{-y}$$

$\therefore\ P(Y_n\le y)\to 1-e^{-y},\ y>0 \quad (\to 0,\ y\le 0) \Rightarrow Y_n \xrightarrow{d}$ exponential (1)

Solution to #9

1. Set $\bar{F}(x) = P(X > x)$. Then

$$P(X > s+t \mid X > s) = P(X > t)$$

$$\Rightarrow P(X > s+t) = P(X > s)\,P(X > t)$$

$$\Rightarrow \bar{F}(s+t) = \bar{F}(s)\,\bar{F}(t)$$

$$\Rightarrow \bar{F}(t_1 + \cdots + t_n) = \bar{F}(t_1)\cdots\bar{F}(t_n) \qquad \text{-induction}$$

$$\Rightarrow \bar{F}(1) = \bar{F}(\tfrac{1}{n})\cdots\bar{F}(\tfrac{1}{n}) = \left(\bar{F}(\tfrac{1}{n})\right)^n \qquad (*)$$

Since $\bar{F}(0) = 1$ and $\bar{F}$ is right cts we get $\bar{F}(1) > 0$ (since $\bar{F}(\frac{1}{n})$ is close to 1 for large $n$).

Let $r = \frac{m}{n}$ be a rational $> 0$. Then

$$\bar{F}\left(\tfrac{m}{n}\right) = \bar{F}\left(\underbrace{\tfrac{1}{n} + \cdots + \tfrac{1}{n}}_{m \text{ of these}}\right) = \left(\bar{F}(\tfrac{1}{n})\right)^m$$

$$\Rightarrow \bar{F}\left(\tfrac{m}{n}\right) = \left[\bar{F}(1)\right]^{m/n} \quad \text{— by } (*)\ \bar{F}(\tfrac{1}{n}) = [\bar{F}(1)]^{1/n}$$

So $\bar{F}(r) = [\bar{F}(1)]^r$. If $x \geq 0 \in \mathbb{R}$ then

$$\bar{F}(x) = \lim_{r \downarrow x} \bar{F}(r) = \bar{F}(1)^x.$$

Note $\bar{F}(1) \neq 1$ since $\bar{F}(x) \to 0$ as $x \to \infty$.

Set $\lambda = -\log[\bar{F}(1)]$. Then $\lambda > 0$ and

$$\bar{F}(x) = e^{-\lambda x}, \quad x \geq 0$$

$\Rightarrow X \sim \text{exponential}(\lambda)$

**Remark**: In the above solution F(1) can't be 1 since X is not a constant.