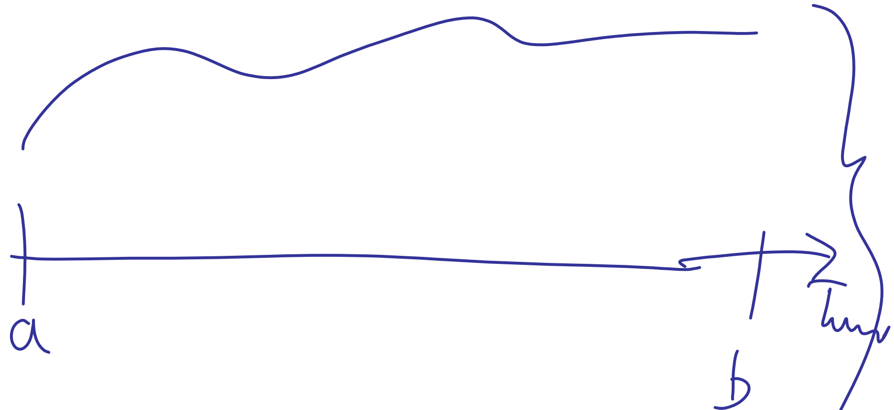
$P(A)$ — Axioms

$$E(X) = \int X \, dP$$

$$\Omega = \left\{ \right.$$



$$\left. \right\}$$

sample space

sample pt (realization) $\{\omega\} = \Omega$

$$\mathbb{R}, \quad \mathbb{R}^k, \quad Z^k$$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

f'ns : $\Omega \longrightarrow \mathbb{R}$ — rv

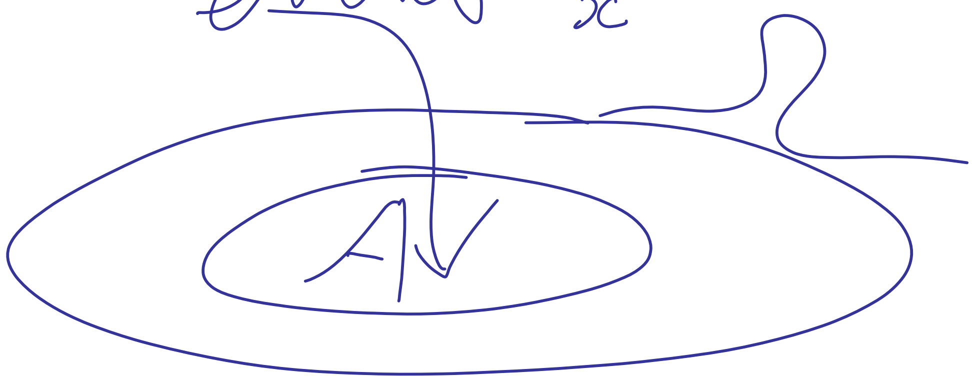
$\qquad\qquad \longrightarrow \mathbb{R}^k$ — rvec

$X, Y, \ldots$ $\qquad \boxed{g(x)}$

$\underset{\sim}{X}, \underset{\sim}{Y}, \ldots$ $\qquad \underbrace{X(\omega)}_{x}$ — #

$$\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}$$

events



I_A or $I(A)$ $\qquad$ <u>imp</u>

$$I(A)(\omega) = 1, \text{ if } \omega \in A$$
$$= 0, \text{ if } \omega \notin A$$

Indicator

Bernoulli

binary

A_1, A_2

Suppose

$$I(A_1) \le I(A_2)$$

$$A_1 \subset A_2$$

$$A_1 \supset A_2$$

$X \le Y$ means

$$X(\omega) \le Y(\omega), \quad \forall \omega$$

X

$X(\omega)$

$$X_1, X_2, \dots; \; X$$

$$X_n \to X \quad \left(X_n(\omega) \to X(\omega), \; \forall \omega \right)$$

$$\underbrace{X_1 \le X_2 \le \cdots \; X_n \to X}$$

$$X_n \uparrow X \qquad X_n \downarrow X$$

$$\boxed{a_n \to a} \quad \boxed{\lim_{n \to \infty} a_n = a}$$

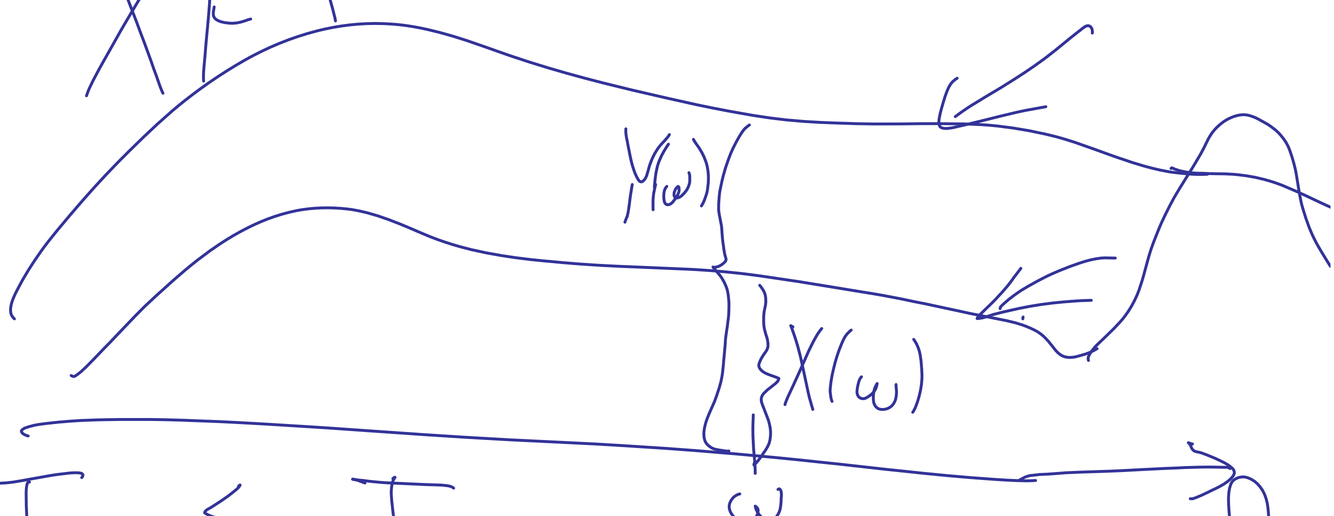
X $\qquad$ $X \le Y$

$X(\omega)$ $\quad$ <u>means</u> $\quad X(\omega) \le Y(\omega), \; \forall \omega \in \Omega$

$X_1 \le X_2 \le X_3 \le \dots$

$X_n \to X \qquad \lim_{n\to\infty} X_n(\omega) = X(\omega), \; \forall \omega \in \Omega$

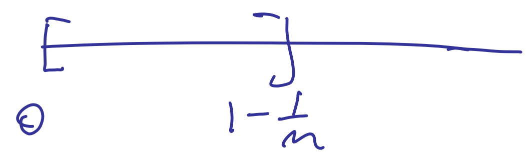
$X \le Y$



$I_{A_1} \le I_{A_2}$

$A_1, A_2, \ldots$

events



$[0, 1+\frac{1}{n}]$, $n = 1, 2, \ldots$ $\longrightarrow [0, 1]$

$\longrightarrow [0, 1)$

Def'n $A_n \to A$ iff $(I(A_n) \to I(A))$

Problem (i) $A_1 \subset A_2 \subset \cdots$ Then

$$\lim_{n\to\infty} A_n = \bigcup_{k=1}^{\infty} A_k$$

(ii) $A_1 \supset A_2 \supset \cdots$ Then

$$A_n \longrightarrow \bigcap_{k=1}^{\infty} A_k$$

rv's X $\qquad E(X)$

Axioms

$1(\omega) = 1$

1. $E(1) = 1$ — E is normed

2. $X \geq 0 \Rightarrow E(X) \geq 0$ — positive property

3. $E(cX + dY) = c\,E(X) + d\,E(Y)$ — linear

4. $X_n \uparrow X$ then $E(X_n) \to E(X)$

4' $0 \leq X_n \uparrow X \Rightarrow E(X_n) \to E(X)$ $\quad$ (MCT)

Prop $1, 2, 3, 4 \iff 1, 2, 3, 4' \iff 1, 2, 3, 4''$

4'' $E\left(\sum_{k=0}^{\infty} X_k\right) = \sum_{k=0}^{\infty} E(X_k)$

$X_k \geq 0$

Def'n $A \subset \Omega$ then

$$P(A) = E[I(A)]$$

$A_1, A_2 \leftarrow \underbrace{\text{don't overlap}}_{\text{disjoint}}$

$$I(A_1 \underset{\text{or}}{\cup} A_2) = I(A_1) + I(A_2)$$

$A_1, A_2, \dots$ don't overlap

$$\Rightarrow I\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} I(A_k)$$

obvious

In general $\boxed{I\left(\bigcap_{k=1}^{\infty} A_k\right) = \prod_{k=1}^{\infty} I(A_k)}$

Sol'n Let $\omega \in \bigcup_{k=1}^{\infty} A_k$

$$\Rightarrow I\left(\bigcup_{k=1}^{\infty} A_k\right)(\omega) = 1$$

Also $\omega \in \bigcup_{k=1}^{\infty} A_k \Rightarrow \omega$ is in at least one of the A_k's

$\Rightarrow \omega$ is in exactly one of the A_k's

($\because$ the A's don't overlap)

$$\Rightarrow \sum_{k=1}^{\infty} I(A_k)(\omega) = 1$$

If $\omega \notin \bigcup_{k=1}^{\infty} A_k \Rightarrow \omega$ is not in any of the A_k's

$$\therefore I\left(\bigcup_{k=1}^{\infty} A_k\right)(\omega) = 0 \;\&\; \sum_{k=1}^{\infty} I(A_k)(\omega) = 0$$

$$\therefore I\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} I(A_k)$$

Show $\rightarrow I(A \cup B) = I(A) + I(B) - I(AB)$

Note: $I(\underbrace{A \cup A^c}_{\substack{\Omega \\ \| \\ 1}}) = I(A) + I(A^c)$

Proposition (Kolmogorov Axioms for P)

(i) $P(\Omega) = 1$

(ii) $P(A) \geq 0$

(iii) $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

$A_1 A_2 = \emptyset$

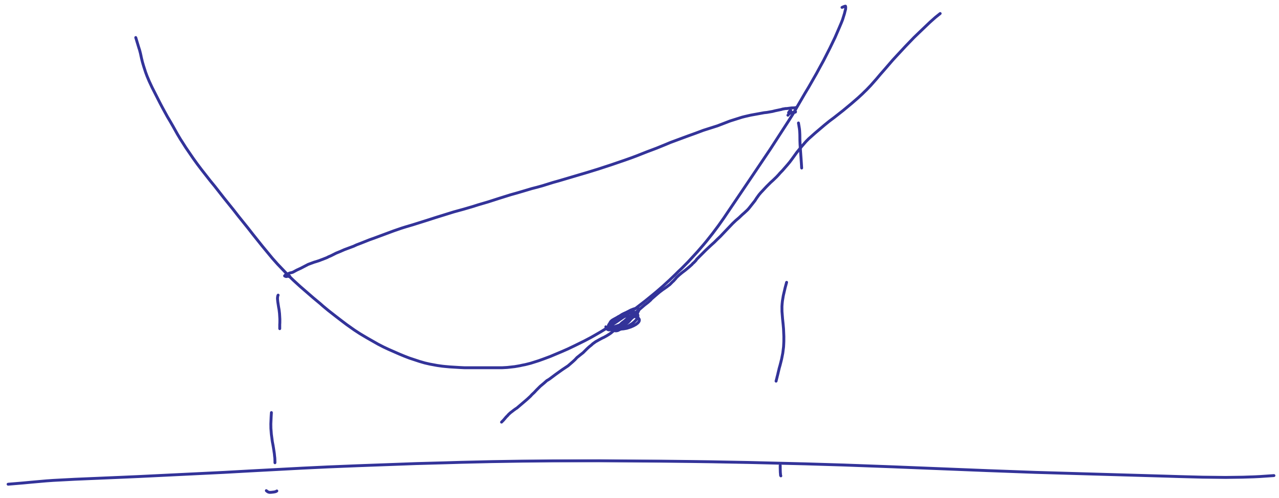
empty or impossible event

(iv) If $A_1, A_2, \ldots$ don't overlap then

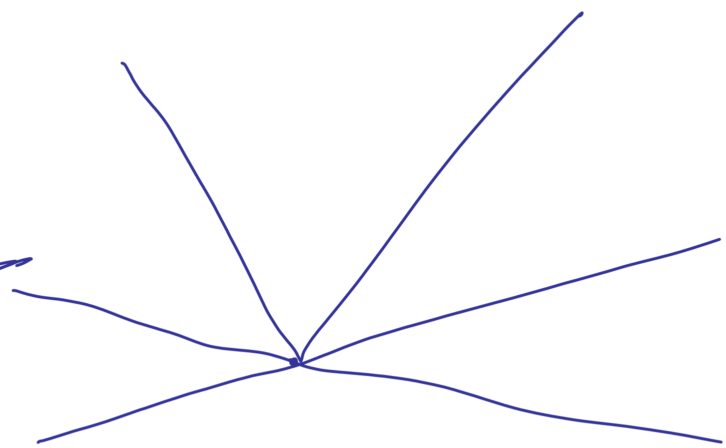
$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

"4"

Proof Do it



convex



For any x_0 $\exists$ a # c s.t. (such that)

there exists

$$g(x) \geq g(x_0) + c(x - x_0), \ \forall x$$

$\parallel$

$g'(x_0)$ if it exists

Prop $X \le Y \Rightarrow E(X) \le E(Y)$

Proof $X \le Y \Rightarrow Y - X \ge 0$

$\Rightarrow E(Y-X) \ge 0$

$\Rightarrow E(Y) - E(X) \ge 0$

$\Rightarrow E(Y) \ge E(X)$

convex

X, g + look at $g \circ X = g(X)$ gcd

$$\boxed{E[g(X)] \ge g(E(X))}$$

Jensen's Inequality

$$\boxed{P(|X| \ge c) \le \frac{E(|X|)}{c}}$$

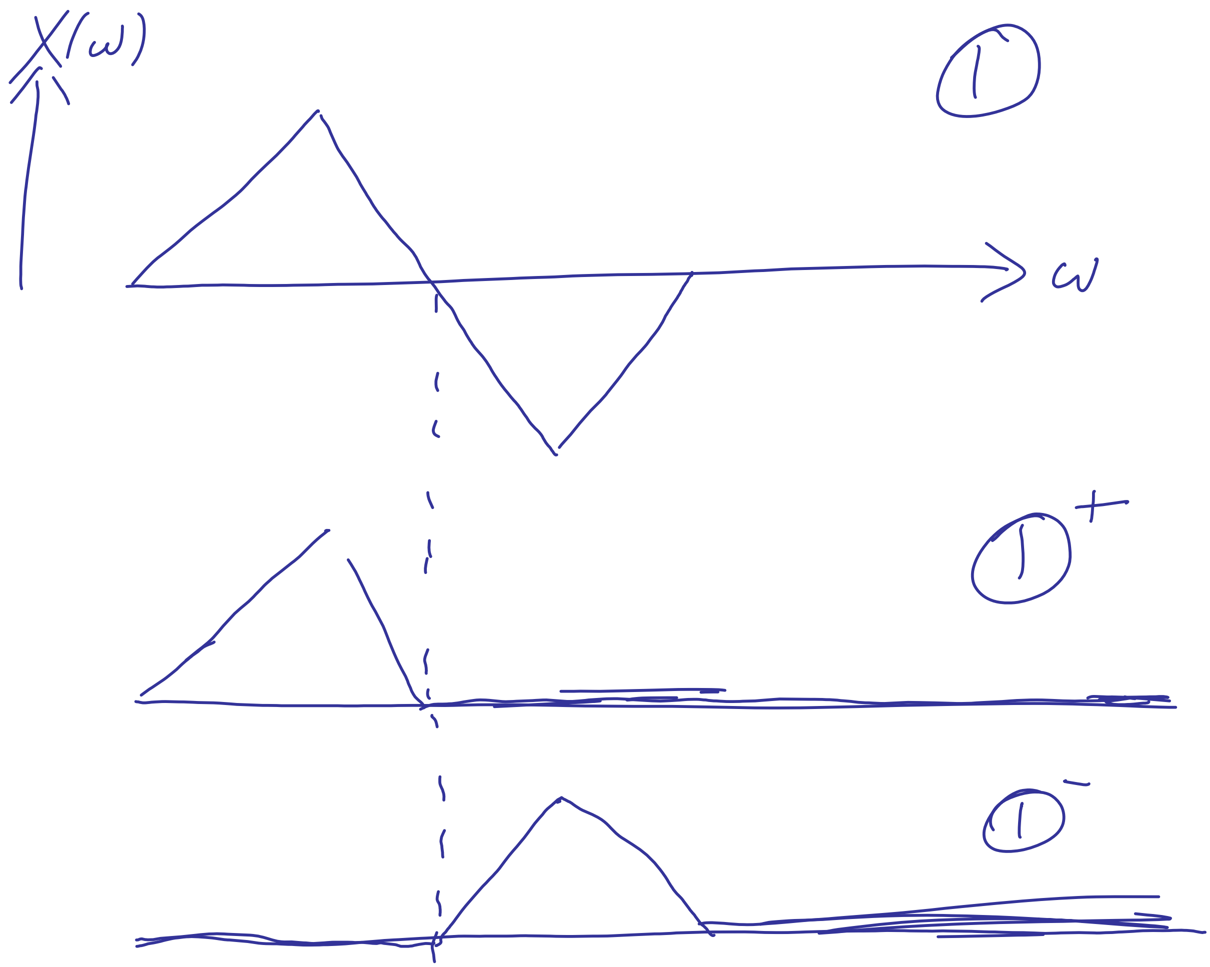
Markov's Inequality

Proof on web " + next week "

$$\left(\bigcup_k A_k\right)^c = \bigcap_k A_k^c$$

$$\left(\bigcap_k A_k\right)^c = \bigcup_k A_k^c$$

de Morgan



$$X = X^+ - X^-$$

$$\boxed{E(|X|) < \infty}$$

$$|X| = X^+ + X^-$$

$$E(X) = E(X^+) - E(X^-)$$