Recall

$X_1 \sim \text{gamma}(r_1)$ ⟵ ind

$X_2 \sim \text{gamma}(r_2)$ ⟵

pdf for X_i is

$$f(x_i) = \frac{x_i^{r_i - 1} e^{-x_i}}{\Gamma(r_i)}, \quad x_i > 0$$

The mgf for X_i is

$$m_i(t) = \left(\frac{1}{1-t}\right)^{r_i}, \quad t < 1$$

The mgf of $Y = X_1 + X_2$ is

$$m_Y(t) = \left(\frac{1}{1-t}\right)^{r_1 + r_2}, \quad t < 1$$

$\Rightarrow Y \sim \text{gamma}(r_1 + r_2)$

$$\Rightarrow f(y) = \frac{y^{r_1 + r_2 - 1} e^{-y}}{\Gamma(r_1 + r_2)}, \quad y > 0$$

Look at

$$Y_1 = \frac{X_1}{X_1 + X_2} \quad \text{is a beta rv}$$

$$Y_2 = X_1 + X_2 \qquad \Big(\overbrace{Y_2 = X_1}^{\text{easier}}\Big)$$

Now we have a 1-1 f'n from $\underset{\sim}{X}$ to $\underset{\sim}{Y}$

$$\Rightarrow f_{\underset{\sim}{Y}}(y) = f_{\underset{\sim}{X}}(\underset{\sim}{x} \text{ in terms of } y) \left| \det\left(\frac{dx}{dy'}\right) \right|$$

This yields

$$f(y_1, y_2)$$

$$\Rightarrow f(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2$$

& done.

$$y_1 = \frac{x_1}{x_1+x_2}$$

$$y_2 = x_1+x_2$$

$$\Rightarrow \quad x_1 = y_1 y_2$$

$$x_2 = y_2 - y_1 y_2$$

$$\frac{d\underset{\sim}{x}}{d\underset{\sim}{y}'} = \begin{pmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{pmatrix}$$

$$\Rightarrow \det\left(\frac{d\underset{\sim}{x}}{d\underset{\sim}{y}'}\right) = y_2$$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

$$= \frac{x_1^{r_1-1} e^{-x_1}}{\Gamma(r_1)} \frac{x_2^{r_2-1} e^{-x_2}}{\Gamma(r_2)}$$

$$f_{\underset{\sim}{X}}(y_1y_2, y_2-y_1y_2) = \frac{(y_1y_2)^{r_1-1} e^{-y_1y_2}\, y_2^{r_2-1}(1-y_1)^{r_2-1} e^{-y_2+y_1y_2}}{\Gamma(r_1)\Gamma(r_2)}$$

$$= y_1^{r_1-1}(1-y_1)^{r_2-1}\, y_2^{r_1+r_2-2} e^{-y_2} \Big/ \Gamma(r_1)\Gamma(r_2)$$

$$\Rightarrow f(y_1, y_2) = \frac{y_1^{r_1-1}(1-y_1)^{r_2-1}\,\Gamma(r_1+r_2)}{\Gamma(r_1)\,\Gamma(r_2)} \quad \frac{y_2^{r_1+r_2-1} e^{-y_2}}{\Gamma(r_1+r_2)}$$

$$\Rightarrow f(y_1) = \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\,\Gamma(r_2)}\, y_1^{r_1-1}(1-y_1)^{r_2-1}, \quad 0 < y_1 < 1$$

Note - $r_1 = r_2 = 1 \longrightarrow$ uniform $(0, 1)$

- $\frac{X_1}{X_1+X_2}$ & X_1+X_2 are independent

Set $\quad Y = \frac{X_1}{X_1+X_2}$

$$\Rightarrow Y(X_1+X_2) = X_1$$

$$\Rightarrow E(Y)\,E(X_1+X_2) = E(X_1) \quad \text{, by ind}$$

$$\Rightarrow E(Y) = \frac{E(X_1)}{E(X_1+X_2)} = \frac{r_1}{r_1+r_2}$$

Some little facts

X, pdf $f(x)$; $Y = h(X)$; pdf of Y?

$Y = X + b$ pdf ✓ $\qquad Y = \frac{1}{1+X}$ pdf ✓

$Y = \underset{\substack{\uparrow \\ >0}}{c} X$ pdf ✓

$Y = \frac{1}{X}$ pdf ✓

$X_1 \sim \chi^2(m_1)$ ← ind

$X_2 \sim \chi^2(m_2)$ ←

$$Y = \frac{X_1/m_1}{X_2/m_2} \sim F(m_1, m_2)$$

pdf of Y?

$$Y = c\,\frac{X_1}{X_2}$$

$\Rightarrow$ know if we can get the pdf of

$$W = \frac{X_1}{X_2}$$

Now

$$\frac{X_1}{X_2} + 1 = \frac{X_1 + X_2}{X_2}$$

& so *if* we can get the pdf of

$$\frac{X_1 + X_2}{X_2}$$

Then we can get the pdf $\frac{X_1}{X_2}$ & then of Y. But

$$\frac{X_1 + X_2}{X_2} = \frac{1}{\frac{X_2}{X_1 + X_2}} \leftarrow \text{beta}$$

& we are done!

eg Z_1, Z_2 iid $N(0,1)$

$$Y_1 = \frac{Z_1}{Z_2} \quad ; \quad \left.\begin{array}{l} y_1 = \frac{z_1}{z_2} \\ y_2 = z_2 \end{array}\right\} \Rightarrow \begin{array}{l} z_1 = y_1 y_2 \\ z_2 = y_2 \end{array}$$

$$\Downarrow$$

$$\frac{d\underset{\sim}{z}}{d\underset{\sim}{y}'} = \begin{pmatrix} y_2 & y_1 \\ 0 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\det(\ ) = y_2$$

$$Y_2 = Z_2$$

+ so

$$f(z_1, z_2) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-(z_1^2 + z_2^2)/2}$$

$$f_{\underset{\sim}{Y}}(y_1, y_2) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-(y_1^2 y_2^2 + y_2^2)/2} \, |y_2|$$

$$\Rightarrow f(y_1) = \frac{1}{\pi} \int_0^\infty e^{-y_2^2(1+y_1^2)/2} \, y_2 \, dy_2$$

$$= \frac{1}{\pi(1+y_1^2)}$$

long tails

Cauchy

If $Y \sim$ Cauchy then

$$E(|Y|) = \int_{-\infty}^{\infty} |y| \frac{1}{\pi(1+y^2)} \, dy = \infty$$

Cauchy has no moments.

$$E\left(e^{\overset{\sqrt{-1}}{i} t Y}\right) = e^{-|t|}$$

$$\parallel$$

$$\int_{-\infty}^{\infty} (\cos ty) \frac{1}{\pi(1+y^2)} \, dy$$

If $Y_1, \ldots, Y_n$ are iid Cauchy then

$$\bar{Y} = \frac{Y_1 + \cdots + Y_n}{n} \quad \& \quad E\left(e^{it\bar{Y}}\right) = e^{-|t|}$$

$N(\underset{\sim}{\mu}, \Sigma)$

$$\Sigma = \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} = T T' \; ; \; \underset{m\times 1}{\underset{\sim}{Z}} \sim N(\underset{\sim}{0}, I) ; \; f_{\underset{\sim}{Z}}(\underset{\sim}{z}) \propto e^{-|\underset{\sim}{z}|^2/2}$$

$$\underset{\sim}{Y} = \underset{\sim}{\mu} + T \underset{\sim}{Z}$$

$$\underset{\sim}{z} = T^{-1}(y - \underset{\sim}{\mu}) \Longrightarrow \frac{\partial \underset{\sim}{z}}{\partial y'} = T^{-1} \Longrightarrow \det\left(\frac{\partial \underset{\sim}{z}}{\partial y'}\right) = \frac{1}{\det T} = \frac{1}{\sqrt{\det \Sigma}}$$

$$\Longrightarrow f_{\underset{\sim}{Y}}(y) = \frac{1}{\sqrt{\det(\Sigma)}} \left(\frac{1}{\sqrt{2\pi}}\right)^m \exp\left\{- \underbrace{[T^{-1}(y-\underset{\sim}{\mu})]'[T^{-1}(y-\underset{\sim}{\mu})]}_{(y-\underset{\sim}{\mu})' \Sigma^{-1} (y-\underset{\sim}{\mu})}/2]\right\}$$

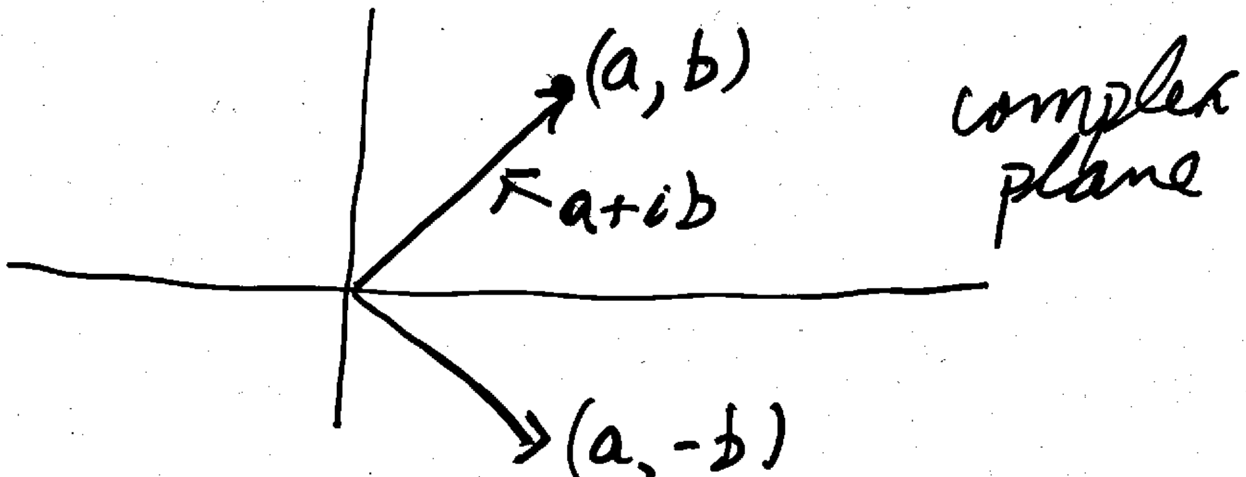
Note: 1 Σ diagonal $\Longrightarrow f_{\underset{\sim}{Y}}(y)$ factors out

$$2 \; m_{\underset{\sim}{Y}}(\underset{\sim}{t}) = E(e^{\underset{\sim}{t}'\underset{\sim}{Y}}) = e^{\underset{\sim}{t}'\underset{\sim}{\mu}} E(e^{\overbrace{\underset{\sim}{t}'T}^{\text{row-vector}} \underset{\sim}{Z}}) = e^{\underset{\sim}{t}'\underset{\sim}{\mu}} e^{\underset{\sim}{t}'\Sigma \underset{\sim}{t}/2}$$

$$3 \; c_{\underset{\sim}{Y}}(\underset{\sim}{t}) = m_{\underset{\sim}{Y}}(i\underset{\sim}{t}) = e^{i\underset{\sim}{t}'\underset{\sim}{\mu}} e^{-\underset{\sim}{t}'\Sigma \underset{\sim}{t}/2}$$

Some background

$a, b \in \mathbb{R}$ & i an object st $i^2 = -1$ then $\underset{\substack{\uparrow \\ \text{real} \\ \text{part}}}{a} + i\underset{\substack{\uparrow \\ \text{imaginary} \\ \text{part}}}{b}$ is called a complex #.



complex conjugate $\overline{a+ib} = a - ib$

If $z = a + ib$ then $|z|^2 = a^2 + b^2$

Notice $z\bar{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$

$e^{it} = \cos t + i \sin t$ (def'n)

Note $e^z = e^a e^{ib}$

$$\overline{e^{it}} = \cos t - i \sin t$$

$$= \cos(-t) + i \sin(-t) = e^{-it}$$

$$e^{it_1} e^{it_2} = e^{i(t_1+t_2)}$$ - can check

$$|e^{it}|^2 = \cos^2 t + \sin^2 t = 1$$

$$\Rightarrow |e^{it}| = 1$$

$$|e^{it_2} - e^{it_1}| \leq |t_2 - t_1|$$

$$|e^{it_2} - e^{it_1}| \leq 2$$

Note $|z_1 + z_2| \leq |z_1| + |z_2|$

$$|e^{it_2} - e^{it_1}| = \left| \int_{t_1}^{t_2} \frac{d(e^{it})}{dt} \, dt \right|$$

$$\leq \left| \int_{t_1}^{t_2} \left| \frac{de^{it}}{dt} \right| dt \right|$$

$$= \left| \int_{t_1}^{t_2} \underbrace{|ie^{it}|}_{1} \, dt \right| = |t_2 - t_1|$$

Note 1. $|z_1 z_2| = |z_1|\,|z_2|$

2. $|\cos t_2 - \cos t_1| \leq |t_2 - t_1|$

$|\sin t_2 - \sin t_1| \leq |t_2 - t_1|$

X rv

$$c(t) = E\left(e^{itX}\right) \quad \text{---} \quad \text{characteristic f'n (c.f.)}$$

$$= E[\cos(tX)] + i\,E[\sin(tX)]$$

Proposition

(i) $c(0) = 1$, $|c(t)| \leq 1$

(ii) $\overline{c(t)} = c(-t)$

(iii) $c_{aX+b}(t) = e^{itb} c_X(at)$

Inversion Theorem

(i) If X has pdf $f(x)$ then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\, c(t)\,dt$$

$$\left[\; c(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx \;\right]$$

(ii) $$F(y) - F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{-ity} - e^{-itx}}{-it}\right) c(t)\,dt$$

at ct pts of F

Note:- $c(t)$ determines the dist'n

$\exists\ \underset{\sim}{X}$ $\quad c(\underset{\sim}{t}) = E\left(e^{i\underset{\sim}{t}'\underset{\sim}{X}}\right)$

$\exists\ c(\underset{\sim}{t}) \longleftrightarrow F(\underset{\sim}{x}) = P(\underset{\sim}{X} \le \underset{\sim}{x})$

Fact $F(\underset{\sim}{x}) = F(x_1)\cdots F(x_n) \Leftrightarrow$ independence

$c(\underset{\sim}{t}) = c(t_1)\cdots c(t_n) \Rightarrow F(\underset{\sim}{x}) = F(x_1)\cdots F(x_n) \Rightarrow$ ind

Prop X & Y ind $\Rightarrow c_{X+Y}(t) = c_X(t)\,c_Y(t)$

Pf $c_{X+Y}(t) = E\left[e^{it(X+Y)}\right]$

$$= E\left(e^{itX}e^{itY}\right)$$

$$= E\left(e^{itX}\right)E\left(e^{itY}\right)$$

Note $c_{X+Y}(t) = c_X(t)\,c_Y(t) \not\Rightarrow$ ind

eg X is Cauchy if

$$f(x) = \frac{1}{\pi(1+x^2)}$$

$$c(t) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi(1+x^2)}\,dx = e^{-|t|}$$

Let $Y = X$. Then Y & X are not ind.

$$c_{X+Y}(t) = c_{2X}(t) = c_X(2t) = e^{-2|t|}$$

$$c_X(t)\, c_Y(t) = e^{-|t|} e^{-|t|} = e^{-2|t|}$$

& so $C_{X+Y}(t) = C_Y(t)\, C_Y(t)$

but Y & X are dependent.

<u>Remarks</u>

- $m(t) = E(e^{tX})$ — mgf
- $m(\underset{\sim}{t}) = E(e^{\underset{\sim}{t}'\underset{\sim}{X}})$ — mgf of $\underset{\sim}{X}$

<u>Fact</u>: If $m(\underset{\sim}{t})$ exists in a neighbourhood of $\underset{\sim}{0}$ then $c(\underset{\sim}{t}) = m(i\underset{\sim}{t})$

eg $X \sim N(0,1) \Leftrightarrow m(t) = e^{t^2/2}, \ \forall t$

$$\Rightarrow c(t) = e^{(it)^2/2} = e^{-t^2/2}$$

3 A version of Taylor's Theorem

If $g^{(m)}(0)$ exists then

$$g(x) = \left[\sum_{j=0}^{m} \frac{g^{(j)}(0)}{j!} x^j\right] + o(x^m),$$

as $x \to 0$. Here $o(x^m)$ is the remainder & has the property that

$$\lim_{x \to 0} \frac{o(x^m)}{x^m} = 0$$

Also, $g^{(0)} = g$.

eg $e^x = 1 + x + o(x)$

$$e^{x + o(x)} = 1 + x + o(x)$$

not the same expression!

Formally,

$$e^{itX} = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} X^j$$

"& so" $$E(e^{itX}) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} E(X^j)$$

"& so" $$c^{(j)}(0) = i^j E(X^j)$$

Note, $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$

& we "know" $m^{(j)}(0) = E(X^j)$

Theorem Let X be finite wp1. Then $c(t)$ is uniformly cts. If $E(|X|^k) < \infty$ then $E(X^k e^{itX})$ is uniformly cts (in t).

Proof Let $\epsilon > 0$. Then $|c(t_2) - c(t_1)|$

$$= |E(e^{it_2 X} - e^{it_1 X})|$$

$$\le E(|e^{it_2X} - e^{it_1X}|)$$

Now choose A so that

$$2P(|X| \geqslant A) \le \epsilon/2 .$$

Now

$$E(|e^{it_2X} - e^{it_1X}|)$$

$$= E(\overbrace{|\qquad\qquad|}^{\le |t_2-t_1|\overbrace{|X|}^{\le A}} I(|X|<A))$$

$$+ E(\underbrace{|\qquad|}_{\le 2} I(|X| \geqslant A))$$

$$\le A|t_2-t_1| + 2P(|X| \geqslant A)$$

$$\le A|t_2-t_1| + \epsilon/2$$

$$\therefore |t_2-t_1| \le \frac{\epsilon}{2A} \Rightarrow |c(t_2) - c(t_1)| \le \epsilon$$

qed

Consequence

① $E(|X|^k) < \infty \Rightarrow c^{(k)}(0)$ exists

(try to show for $k=1$ by def'n of a derivative)

$c^{(k)}(0)$ exists $\Rightarrow E(|X|^k) < \infty$?

No unless k is even.

② $E(|X|^k) < \infty$

$$\Rightarrow c(t) = \sum_{j=0}^{k} \frac{(it)^j}{j!} E(X^j) + o(t^k)$$

$$\boxed{c^{(k)}(0) = i^k E(X^k)}$$

Some more background

real #'s

set of #'s $\left\{\begin{array}{l}\text{bounded above}\\ \text{bounded below}\end{array}\right\}$ bounded

$\swarrow$

$\exists$ lub = sup

& glb = inf

sequences $a_n \to a$

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k \to a\right)$$

→ has the Cauchy property in

That if $a_n - a_m \to 0$ as mutual conv

$n+m \to \infty$ then $\exists$ an a st

$$a_n \to a$$

Types of convergence

1 $X_n \xrightarrow{ms} X$ (in mean square) if

$$E(X_n - X)^2 \to 0 \quad \text{as } n \to \infty$$

Fact $\xrightarrow{ms}$ has the Cauchy property

2 $X_n \xrightarrow{P} X$ (convergence in probability)

$\forall \epsilon > 0$

$$P(|X_n - X| \le \epsilon) \to 1 \quad \text{as } n \to \infty$$

Fact $\xrightarrow{P}$ has the Cauchy property

3 $X_n \longrightarrow X$ if $\lim_{n \to \infty} X_n(\omega) = X(\omega)$,

$\forall \omega \in \Omega$.

almost surely

4 $X_n \xrightarrow{wp1} X$ $(X_n \xrightarrow{as} X)$ if

$$P(X_n \to X) = 1$$

Theorem If $0 \le X_n$ & $X_n \uparrow X$ wp1 then $E(X_n) \to E(X)$.

Monotone convergence theorem (MCT)

Note $0 \le X_n$ can be replaced by assuming $E(|X_1|) < \infty$.

Theorem (DCT) If $X_n \xrightarrow{as} X$ & $|X_n| \le Y$ with $E(Y) < \infty$ then

$$E(X_n) \to E(X)$$

Note $X \overset{as}{=} X' \Rightarrow E(X) = E(X')$ which allows $X_n \uparrow X$ to be replaced by $X_n \uparrow X$, wp1. (Why?)

Another type of convergence

Def'n X_n converges to X weakly if $\forall$ bounded cts h we have

$$E(h(X_n)) \to E(h(X)) \quad (*)$$

Note — also called convergence in dist'n

— $X_n \xrightarrow{d} X$

separating class of h's (subset of all bd'd cts f'ns such that if $(*)$ holds for them it holds for all bd'd cts f'ns)

$$1 \quad \underbrace{\{\sin tx \,, \cos tx \mid \forall t\}}_{\{e^{itx} \mid \forall t\}}$$

ie. if $c_n(t) \to c(t)$, $\forall t$

then $X_n \xrightarrow{d} X$.

Weak Law of Large Numbers (WLLN)

Let $X_1, X_2, \dots$ be iid with mean μ. Then

$$\bar{X} \xrightarrow{d} \mu$$

Note $Y_n \xrightarrow{d} c \Rightarrow Y_n \xrightarrow{P} c$ $\}$ problem

Proof: Let $c(t)$ be the cf of X_1. We know

$$c(t) = 1 + i\mu t + o(t) = e^{i\mu t + o(t)}$$

So
$$E\left(e^{it\overline{X}}\right) = E\left(e^{i\frac{t}{n}(X_1+\cdots+X_n)}\right)$$
$$= E\left(e^{i\frac{t}{n}X_1}\right)\cdots E\left(e^{i\frac{t}{n}X_n}\right)$$
$$= \left(c\left(\tfrac{t}{n}\right)\right)^n$$
$$= \left[e^{i\mu\frac{t}{n} + o\left(\frac{t}{n}\right)}\right]^n$$
$$= e^{i\mu t + n\,o\left(\frac{t}{n}\right)}$$
$$= e^{i\mu t + o\left(\frac{t}{n}\right)/\left(\frac{1}{n}\right)}$$
$$\to e^{i\mu t}$$
which is the cf of the constant μ.

qed

Central Limit Theorem

Let $X_1, X_2, \dots$ be iid with mean μ + variance σ^2. Then

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

Proof The cf of X_1 is

$$c(t) = E\left(e^{itX_1}\right)$$

$$= e^{it\mu} E\left(e^{it\overbrace{(X_1-\mu)}^{\substack{\text{2nd moment}=\sigma^2 \\ \text{mean}=0}}}\right)$$

$$= e^{it\mu}\left(1 - \frac{\sigma^2 t^2}{2} + o(t^2)\right)$$

$$= e^{it\mu} e^{-\sigma^2 t^2/2 + o(t^2)}$$

$$= e^{it\mu - \sigma^2 t^2/2 + o(t^2)}$$

$$E\left(e^{it\frac{\bar{X}_1 - \mu}{\sigma/\sqrt{n}}}\right)$$

$$= E\left(e^{i\frac{t}{\sigma\sqrt{n}}[(X_1-\mu)+\cdots+(X_n-\mu)]}\right)$$

$$= \left(E\left(e^{i\frac{t}{\sigma\sqrt{n}}(X_1-\mu)}\right)\right)^n$$

$$= \left(e^{-\sigma^2\frac{t^2}{2\sigma^2 n}+o\left(\frac{t^2}{\sigma^2 n}\right)}\right)^n$$

$$= e^{-t^2/2 + n\,o\left(\frac{t^2}{\sigma^2 n}\right)}$$

$$\to e^{-t^2/2}$$

qed

<u>Theorem (DCT)</u> Suppose $X_n \to X$ & $|X_n| \le W$ with $E(W) < \infty$. Then

$$E(X_n) \to E(X)$$

<u>Note</u> The conditions $X_n \to X$ & $|X_n| \le W$ can be replaced by $X_n \xrightarrow{as} X$ & $|X_n| \overset{as}{\le} W$

<u>Prof</u> Let $Z_n = |X_n - X| \le |X_n| + |X| \le 2W$

Then, setting $Y_n = \sup_{k \ge n} Z_k$, yields

$$0 \le Z_n \le Y_n \le 2W$$

Now $E(Y_1) < \infty$ & $Y_n \downarrow 0 \Rightarrow -Y_n \uparrow 0$ with $E(-Y_1)$ existing. Now use the MCT to get $E(Y_n) \to 0$. Since $0 \le Z_n \le Y_n$ we have $E(Z_n) \to 0$. Finally

$$|E(X_n) - E(X)| \le E|X_n - X| \to 0$$

so that $E(X_n) \to E(X)$

qed

$$\xrightarrow{as}$$

$X_n \xrightarrow{as} X$ if $P(X_n \to X) = 1$ (*)

$\{X_n \to X\} = \{\omega \mid X_n(\omega) \to X(\omega)\}$. Denote this event by D.

An equivalent def'n of $\xrightarrow{as}$ is (**)

$X_n \xrightarrow{as} X$ if $\forall \epsilon > 0$ $P(|X_m - X| \le \epsilon, \forall m \ge n) \to 1$

(ie $\forall \epsilon > 0$ $P(\sup_{m \ge n} |X_m - X| \le \epsilon) \to 1$)

Remark 1 $X_n \xrightarrow{as} X \iff |X_n - X| \xrightarrow{as} 0$, as $n \to \infty$

$\iff \sup_{m \ge n} |X_m - X| \xrightarrow{P} 0$, as $n \to \infty$

2 $X_n \xrightarrow{P} X \iff |X_n - X| \xrightarrow{P} 0$

From 1 & 2 we see that

$$X_n \xrightarrow{as} X \implies X_n \xrightarrow{P} X$$

(*) $\Leftrightarrow$ (**)

Assume (**). The events $\bigcap_{m=n}^{\infty} \{|X_m - X| \leq \epsilon\}$ form an increasing sequence (in n) with limit

$$\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{|X_m - X| \leq \epsilon\}$$

Now let $\epsilon_k \downarrow 0$ and set

$$D_k = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{|X_m - X| \leq \epsilon_k\}$$

We have $P(D_k) = 1$, $\forall k$, and D_k forms a decreasing sequence in k. In fact

$$D_k \downarrow D$$

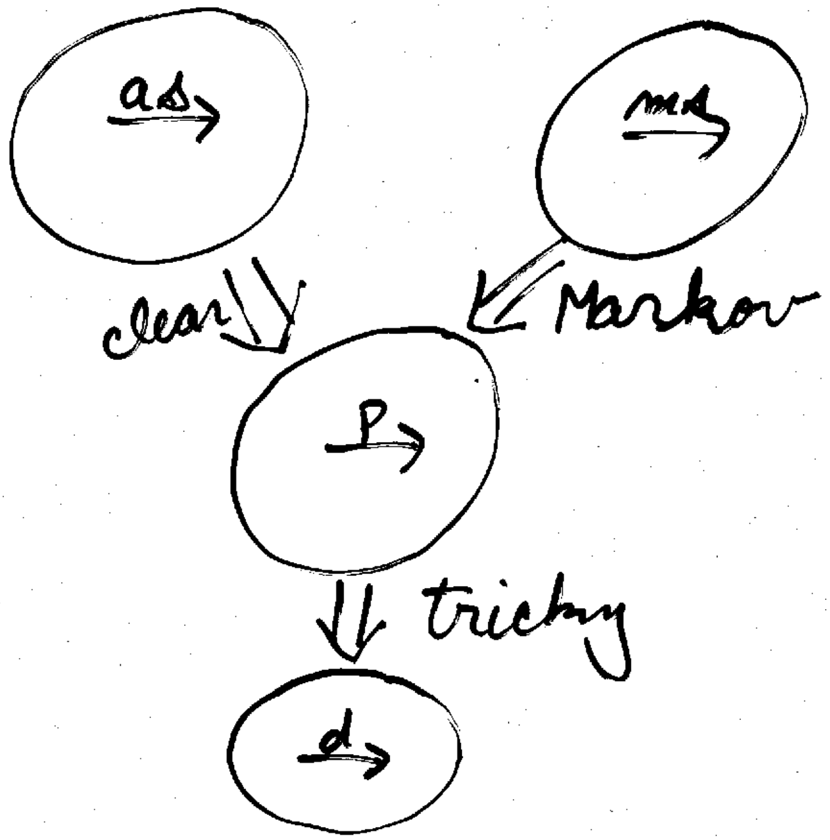
so that $P(D) = 1$ + hence (*) holds.

Assume (*). Then for D_k as above we have

$$1 \geq P(D_k) \geq P(D) = 1$$

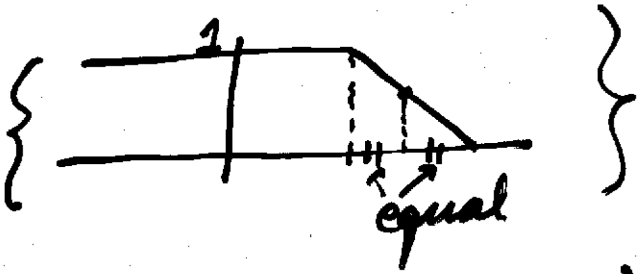
so that $P(D_k) = 1$, $\forall k$ + consequently $\forall \epsilon > 0$

$$P\left(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{|X_m - X| \leq \epsilon\}\right) = 1$$ + so (**) holds.

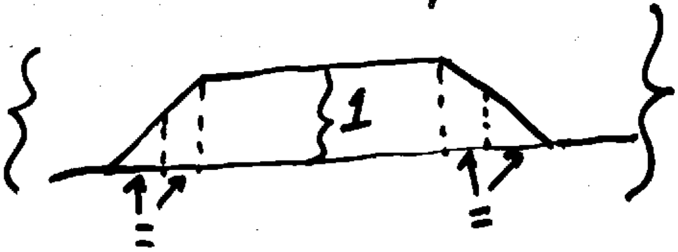


<u>Proposition</u> $X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X$

A separating class



Another



Either of these can be used to show $\xrightarrow{d}$ $\iff F_n(x) \to F(x)$ at continuity points of F

Proposition $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$

Proof Let $\epsilon > 0$. Then

$$\begin{aligned} P(|X_n - c| > \epsilon) &= P(X_n < c-\epsilon) + P(X_n > c+\epsilon) \\ &\le P(X_n \le c-\epsilon) + [1 - P(X_n \le c+\epsilon)] \\ &\to 0 \end{aligned}$$

qed

Little results

$X_n \xrightarrow{d} X \Rightarrow \underset{\text{cts}}{g}(X_n) \xrightarrow{d} g(X)$ — true in the vector case

$X_n \xrightarrow{P} X \Rightarrow \underset{\text{cts}}{g}(X_n) \xrightarrow{P} g(X)$ — "

Subsequence

$X_1, X_2, \cdots$

$n_1 < n_2 < \cdots$ natural #'s

$X_{n_1}, X_{n_2}, \cdots$ is a subsequence

Proposition If $\forall \epsilon > 0 \quad \sum P(|Y_m| > \epsilon) < \infty$ then $Y_m \xrightarrow{as} 0$.

Proof Let $\epsilon > 0$. Then

$$P(|Y_m| \leq \epsilon ; \forall m \geq N)$$
$$= 1 - P\left(\bigcup_{m=N}^{\infty} \{|Y_m| > \epsilon\}\right)$$
$$\overset{\text{Boole}}{\geq} 1 - \sum_{m=N}^{\infty} P(|Y_m| > \epsilon) \longrightarrow 1$$

qed

Corollary1 Let $\epsilon_m \downarrow 0$. Then $\sum P(|Y_m| > \epsilon_m) < \infty$ $\Rightarrow Y_m \xrightarrow{as} 0$

Corollary2 $X_m \xrightarrow{P} X \Rightarrow \exists \; X_{m_k} \xrightarrow{as} X$

Pf of 2 Let $\epsilon_k \downarrow 0$. Choose k st

$$P(|X_{m_k} - X| > \epsilon_k) \leq c_k ,$$

where $\sum c_k < \infty$. The result then follows from Corollary1

Corollary 3 $X_n \xrightarrow{P} X \iff$ every subsequence has a further subsequence converging wp1 to X

Proof Use contradiction.

Recall X_n converges mutually wp1 if $\forall \epsilon > 0$ $P(|X_m - X_n| \le \epsilon, \forall m, n \geq N) \to 1$ as $N \to \infty$. This implies $\exists X$ st $X_n \xrightarrow{as} X$.

Note $X_n \xrightarrow{as} X$ if $P(|X_n - X| \le \epsilon, \forall n \geq N) \to 1$

Proposition Let $\epsilon_n > 0$, $\sum \epsilon_n < \infty$ +

$$\sum P(|X_{n+1} - X_n| > \epsilon_n) < \infty$$

Then $X_n \xrightarrow{as} X$

Proof Let $c_N = \sum_{n=N}^{\infty} \epsilon_n$. Then $c_N \downarrow 0$. So, for any $\epsilon > 0$ $c_N \le \epsilon$ $\forall N$ large enough. Now for such N

$$P(|X_m - X_n| \le \epsilon, \forall m, n \geq N)$$

$$\geq P(|X_m - X_n| \leq c_N \,, \forall m, n \geq N)$$

$$\geq P(|X_{n+1} - X_n| \leq \epsilon_n \,, \forall n \geq N)$$

$$\geq 1 - \sum_{n=N}^{\infty} P(|X_{n+1} - X_n| > \epsilon_n) \to 1$$

Hence $\{X_n\}$ is mutually convergent wp1 so that $\exists X$ st $X_n \xrightarrow{as} X$

qed

<u>PDCT</u> $X_n \xrightarrow{P} X$, $|X_n| \leq W$ with $E(W) < \infty$. Then

$$E(X_n) \to E(X)$$

Pf: Use subsequences

<u>Note</u> In fact $E|X_n - X| \to 0$