$\Omega$ – sample space $= \{\omega\}$

$\omega$ – sample point

$A \subset \Omega$ – event

$A^c = \{\omega : \omega \notin A\}$

$\emptyset = \Omega^c$ = empty or impossible event

$\Omega$ – sure event

$A \cup B$ = $A$ or $B$, $A \cap B = AB = A$ and $B$

$\bigcup_k A_k$ = event that at least one of the $A_k$ occurs

$\bigcap_k A_k$ = event that all the $A_k$ occur

$$\left.\begin{aligned} \left(\bigcup A_k\right)^c &= \bigcap A_k^c \\ \left(\bigcap A_k\right)^c &= \bigcup A_k^c \end{aligned}\right\} \text{deMorgan}$$

$A \subset B$ is equivalent to $\underbrace{A \Rightarrow B}_{A \text{ occurs implies } B \text{ occurs}}$

$\underset{\sim}{X} : \Omega \to \mathbb{R}^k$ rvec

$(k=1 \to \text{rv})$

$I(A)$ — indicator rv defined as

$$I(A)(\omega) = 1 \;, \quad \omega \in A$$
$$\qquad\quad = 0 \;, \quad \omega \notin A$$

<u>Useful results</u>

$I(A \cup B) = I(A) + I(B) - I(AB)$

$I(\bigcap A_k) = \prod_k I(A_k)$

$I(\emptyset) = 0 \;, \; I(\Omega) = 1$

$I(A) + I(A^c) = 1$

$A_1, A_2, \ldots$ disjoint $\Rightarrow I(\bigcup_k A_k) = \sum_k I(A_k)$

$A \subset B \Leftrightarrow I(A) \le I(B)$ — see def'n

<u>Def'n</u> $X \le Y$ if $X(\omega) \le Y(\omega) \;, \; \forall \omega \in \Omega$

Sequence of rv's

$X_1, X_2, \ldots$

Def'n $X_n \to X$ if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$, $\forall \omega$

Notation $X_n \uparrow X$ if

$X_1 \le X_2 \le \cdots$ and $X_n \to X$

$X_n \downarrow X$ if

$X_1 \ge X_2 \ge \cdots$ and $X_n \to X$

Def'n $A_n \to A$ if $I(A_n) \to I(A)$

events

Notation $A_n \uparrow A$ if $A_1 \subset A_2 \subset \cdots$ and $A_n \to A$

$A_n \downarrow A$ if $A_1 \supset A_2 \supset \cdots$ and $A_n \to A$

## E-Axioms

(1) $E(1) = 1$

(2) $X \geq 0 \Rightarrow E(X) \geq 0$

(3) $E(cX + dY) = c\,E(X) + d\,E(Y)$

(4) $X_k \geq 0, \ \forall k$ then

$$E\left(\sum_{k=1}^{\infty} X_k\right) = \sum_{k=1}^{\infty} E(X_k)$$

(4)' $0 \leq X_n \uparrow X \Rightarrow E(X_n) \to E(X)$

Proposition $\{(1), (2), (3), (4)\} \Leftrightarrow \{(1), (2), (3), (4)'\}$

Remark 4' is the Monotone Convergence Theorem (MCT).

Def'n $P(A) = E[I(A)]$

## Proposition (Kolmogorov Axioms)

1. $P(\Omega) = 1$
2. $P(A) \geq 0$
3. $A_1, A_2$ disjoint $\Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2)$
4. $A_1, A_2, \ldots$ disjoint $\Rightarrow$

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

Note 4 is the $\sigma$-additive property of $P$

$P$ is a f'n from events to $\mathbb{R}$

$E$ is a f'n from rv's to $\mathbb{R}$

($\Omega$, collection of events, $P$) is a probability space

**Proposition** $X \le Y \Rightarrow E(X) \le E(Y)$

**Def'n** $g: \mathbb{R} \to \mathbb{R}$ is convex if $\forall x_0$ $\exists$ $c$ such that

$$g(x) \geq g(x_0) + c(x - x_0), \quad \forall x$$

**Jensen's Inequality**

$$E[g(X)] \leq g[E(X)]$$

**Markov's Inequality**

$$P(|X| \geq c) \leq \frac{E(|X|)}{c}$$

($c > 0$ is a constant)