Application

Let R_i, $i = 0, 1, 2, \ldots$ be counting rv's with $E(R_i) \le M < \infty$. Then

$$\sum_{i=0}^{\infty} R_i s^i$$

converges wp1 on $|s| < 1$.

Lemma Let $A_1, A_2, \ldots$ be events with $P(A_i) = 1$, $\forall i$. Then

$$P\left(\bigcap_i A_i\right) = 1$$

Pf of Lemma

Boole

$$P\left[\left(\bigcap_i A_i\right)^c\right] = P\left(\bigcup_i A_i^c\right) \le \sum_i P(A_i^c) = 0$$

We now use this to solve our application.

Let $0 < r < 1$ with r rational. Set

$$X_m = \sum_{i=0}^{m} R_i s^i$$

Then $|X_{m+1} - X_m| = R_{m+1} |s|^{m+1}$

$$\leq R_{m+1} r^{m+1}, \text{ on } |s| \leq r$$

$\therefore$ $E(|X_{m+1} - X_m|) \leq M r^{m+1}$, on $|s| \leq r$

Now let $1 > r_1 > r$. We then have

$$\sum P(|X_{m+1} - X_m| > r_1^{m+1})$$

$$\leq \sum M \left(\frac{r}{r_1}\right)^{m+1} < \infty \quad \left(\begin{smallmatrix}\text{geometric}\\ \text{series}\end{smallmatrix}\right)$$

$\therefore$ $\sum_{i=0}^{\infty} R_i s^i$ converges on $|s| \leq r$

wp1. Let $A_r = \{\omega \mid \underset{\text{for some } |s| \leq r}{\underline{\text{no convergence}}}\}$.

Then $\bigcap_{0<r<1} A_r^c = \{\omega \mid$ convergence on $|s| < 1\}$

Since $P(\bigcap_{0<r<1} A_r^c) = 1$ we conclude

$$P\left(\sum_{i=0}^{\infty} R_i s^i \text{ converges on } |s| < 1\right) = 1$$

The Central Limit Theorem

We have $\log(1+x) = x + o(x)$

$$\Rightarrow 1+x = e^{x+o(x)}$$

$$\Rightarrow 1+x+o(x) = e^{x+o(x)} \qquad (*)$$

$$\Rightarrow 1-\frac{x^2}{2}+o(x^2) = e^{-x^2/2} + o(x^2) \qquad (**)$$

Remarks: 1. (*) leads to the WLLN while (**) is used in the CLT

2. (*) remains true for complex x but we then use $o(|x|)$

The Central Limit Theorem

Let $X_1, X_2, \dots$ be iid with mean μ & variance σ^2.
Then

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \quad , \text{ as } n \to \infty$$

Proof

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \underbrace{\left(\frac{X_j-\mu}{\sigma}\right)}_{Y_j}$$

The Y_j are iid with mean 0 & variance 1 so that the cf of Y_1 satisfies

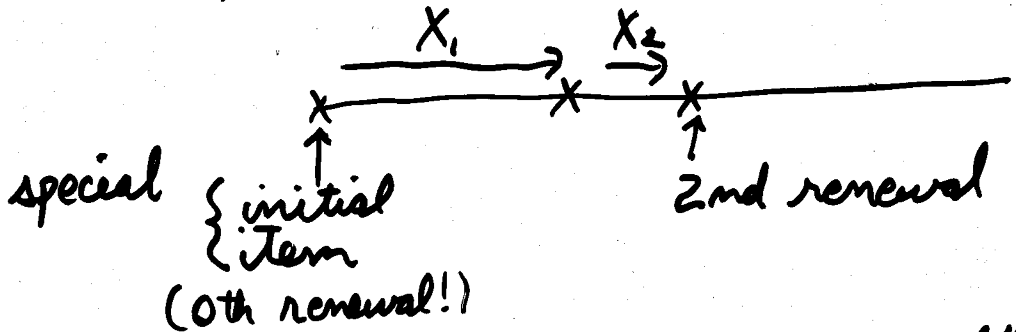
$$c_{Y_1}(t) = 1 - \frac{t^2}{2} + o(t^2) = e^{-\frac{t^2}{2} + o(t^2)}$$

$$\Rightarrow \text{the cf of } \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \left[c_{Y_1}(t/\sqrt{n})\right]^n = e^{-t^2/2 + n\,o(t^2/n)} \to e^{-t^2/2}$$

which is the cf of a $N(0,1)$.

qed

Renewal processes



$X_0 = 0$ (needed to deal with the "renewal" initial item)

$X_1, X_2, \ldots$ iid interarrival times between renewals

$S_r = X_0 + X_1 + \cdots + X_r$ = times to renewals

R_t = # of renewals at time t

$u_t = E(R_t)$

We have

$$\sum_{t \geq 0} R_t z^t = \sum_{r \geq 0} z^{S_r}$$

(Note sums converge)

from which

$$U(z) = \frac{1}{1 - G(z)}$$

where G is the pgf of X_1 & U is the generating function of $u_0, u_1, \ldots$

Remark The above for discrete time.

Notice

$$G(z) = \frac{U(z) - 1}{U(z)}$$

$\Rightarrow$ knowing the renewal probabilities yields the dist'n of the interarrival times.

eg Toss coin $P(H) = p$. When H put a x. The x's form a renewal process. Here

$u_t = P(\text{renewal at time } t) = p$

$(u_0 = 1)$

$$\Rightarrow U(z) = 1 + pz + pz^2 + \dots$$

$$= 1 + \frac{pz}{1-z} \Rightarrow G(z) = \frac{pz}{1-qz}$$

(geometric)

eg Random walk on the integers. Start at 0 move +1 with prob p & -1 with prob q. Let S_t = position at time t. We are interested in returns to 0. We have

$$P(S_{2m}=0) = \binom{2m}{m} p^m q^m$$

& so using Stirling we get

$$\underbrace{\sum_{m=1}^{\infty} P(S_{2m}=0)}_{\text{mean \# of returns to } 0} = \infty \quad \text{iff } p=q$$

$\Rightarrow$ certain return to 0 iff $p=q$

Average time to return? In a bit.

When renewals happen the process begins over. This notion generalizes to that of a regeneration point for a stochastic process $\{X_t\}$. In particular, if for each t_0 X_{t_0} is a regeneration point then we have a <u>Markov process</u>. This is defined in many different (equivalent) forms.

We assume t ~~to~~ be "time".

<u>Def'n①</u> $\{X_t\}$ is Markov if for each t f'ns of the present + future $\{X_s, s \geq t\}$ are independent of functions of the past $\{X_s, s < t\}$ given the present of X_t.

<u>Def'n②</u> $\{X_t\}$ is Markov if "$\forall$" f'ns

$$E(\text{f'n}(\{X_s, s \geq t\}) \mid \{X_s, s \leq t\}) = E(\text{f'n}(\{X_s, s \geq t\}) \mid X_t)$$

<u>Def'n③</u> $\{X_t\}$ is Markov if $\forall$ $t_1 < t_2 < \cdots$

$$E(h(X_{t_{n+1}}) \mid X_{t_n}, X_{t_{n-1}}, \ldots, X_{t_1}) = E(h(X_{t_{n+1}}) \mid X_{t_n}), \text{ "}\forall\text{" } h$$

In the discrete/cts cases this amounts to

$$\underset{\substack{\uparrow \\ x_{t_1}}}{f}(\underset{}{x_1}, \ldots, x_n) = f(x_1) f(x_2 \mid x_1) \cdots f(x_n \mid x_{n-1})$$

$$x_1 \leftarrow x_{t_1},\quad x_n \leftarrow x_{t_n},\quad x_1 \leftarrow x_{t_1},\quad x_2 \leftarrow x_{t_2},\quad x_1 \leftarrow x_{t_1},\quad x_n \leftarrow x_{t_m},\quad x_{n-1} \leftarrow x_{t_{m-1}}$$

If the conditional structure does not change over time then the process is said to be <u>time homogeneous</u>. Thus the conditional dist'n of $X_{t+s} | X_t$ does not change with t. We will assume this to be case.

<u>Note</u> This assumption of time homogeneity of the conditional dist'ns is different than that of <u>stationarity</u> which one finds frequently in time series. There $\{X_t\}$ is stationary if the joint dist'ns of $X_{t_1+t}, \ldots, X_{t_k+t}$ do not change with t.

$\{X_t\}$ – Markov + time homogeneous

state space $= \bigcup_t \{$all possible values of $X_t\}$

we mainly consider countable state spaces + talk of the process as being in state i at time t ("$X_t = i$" means the process is in state i at time t).

Let $X_0, X_1, \cdots$ be rv's & $N \in \{0, 1, \cdots\}$ a counting rv. If $\{N=m\}$ depends only on $X_0, \cdots, X_m$ then we call N a stopping time for the sequence.

Note $\{N \le m\}$ can be used!

Wald's Eq'n

Let $X_0 = 0, X_1, \cdots$ be independent with $X_1, X_2, \cdots$ iid. Let N be a stopping time & $\mu = E(X_1)$. Then

$$E\left(\sum_{m=0}^{N} X_m\right) = \mu\, E(N)$$

Proof Since $\{N < m\} \Leftrightarrow X_m$ is not in the sum we have

$$\sum_{m=0}^{N} X_m = \sum_{m=0}^{\infty} X_m\, I(N \ge m)$$

$$= \sum_{m=1}^{\infty} X_m\, I(N \ge m)$$

$$\Rightarrow E\left(\sum_{m=0}^{N} X_m\right) = E\left(\sum_{m=1}^{\infty} X_m\, I(N \ge m)\right)$$

$$= \sum_{m=1}^{\infty} E[X_m \underbrace{I(N \geq m)}_{f^{\prime}n \text{ of } X_1, \ldots, X_{m-1}}]$$

$$= \sum_{m=1}^{\infty} E(X_m) E(I(N \geq m))$$

$$= \mu \sum_{m=1}^{\infty} P(N \geq m) = \mu E(N)$$

qed

Some examples of stopping times

1. Bernoulli trials (0 or 1). Time to first 1 is a stopping time.

2. X_n iid ± 1 prob $\frac{1}{2}$

$$N = \min\{n : X_1 + \cdots + X_n = 1\}$$

is a stopping time (gamble until ahead!). Can show $P(N \text{ finite}) = 1$

Now look at a renewal process on $t \geq 0$. Call it $\{N(t) : t \geq 0\}$. $N(t) = \#$ of renewals up to time t not including the initial item ($= N_t + 1$ in Whittle). If the interarrival times are exponential(λ) & ind $\Rightarrow$ cts time Markov. (In discrete time we would require iid geometric's).

Set $m(t) = E(N(t))$. This is the <u>renewal function</u>. Call $E(X_i)$ $\mu > 0$ ($\therefore P(X_i = 0) < 1$)

<u>Elementary Renewal Theorem</u> (Th 6.1.2 of Whittle)

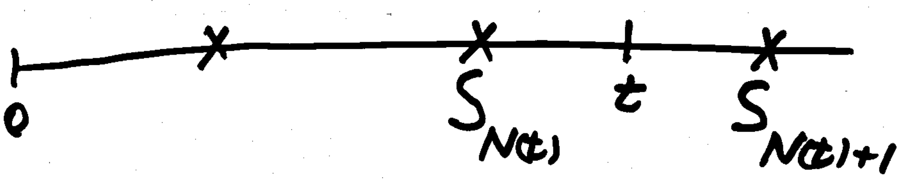
$$\frac{m(t)}{t} \to \frac{1}{\mu}$$

The proof relies on Wald's equation. First note by SLLN

$$\frac{S_n}{n} \xrightarrow{as} \mu \Rightarrow S_n \xrightarrow{wp1} \infty$$

Now since $N(t) \geq n \iff S_n \leq t$ (or $N(t) < n \iff S_n > t$) we must have

$$P(\underset{\underset{t\to\infty}{\lim} N(t)}{\overset{\|}{N(\infty)}} \text{ is finite}) = P(\text{one of the interarrivals is } \infty) \leq \sum_{1}^{\infty} P(X_i = \infty) = 0$$



$$\underset{\underset{\mu}{\downarrow as}}{\frac{S_{N(t)}}{N(t)}} \leq \frac{t}{N(t)} < \underset{\underset{\mu}{\downarrow as}}{\frac{S_{N(t)+1}}{N(t)}} \quad \left(\therefore \frac{N(t)+1}{N(t)} \to 1\right)$$

so that $\frac{N(t)}{t} \xrightarrow{as} \frac{1}{\mu}$.

<u>Proposition</u> $N(t)+1$ is a stopping time for $X_1, X_2, \ldots$

<u>Proof</u>: $N(t)+1 = n \iff N(t) = n-1$

$\iff X_1 + \cdots + X_{n-1} \le t \ \& \ X_1 + \cdots + X_n > t$

Hence $I(N(t)+1=n)$ is a function of $X_1, \ldots, X_n$.

qed

<u>Corollary</u> $E(S_{N(t)+1}) = \underbrace{E(X)}_{\mu} E(N(t)+1)$

This result yields (eq'n 15 p 105)

$$E(N(t)+1) = \frac{E(S_{N(t)+1})}{\mu}$$

$$\approx \frac{t}{\mu}$$ (almost rigorous)

We prove this in 447.

This result has interesting implications for the renewal quantity u_t as

$$E(N(t)+1) = E\left(\sum_{k=0}^{t} R_k\right)$$
$$= \sum_{k=0}^{t} u_k$$

so that

$$\frac{1}{t} \sum_{k=0}^{t} u_k \to \frac{1}{\mu}$$

which <u>almost</u> yields the stronger result

$$u_t \to \frac{1}{\mu} \quad ,$$

a result we show next term. This is the renewal theorem and it has applications to limiting results in Markov processes. For example, if the state space is countable (Markov <u>Chain</u>) and the process is time homogeneous (times between $X_t = i$ are iid) then $X_t = i$ will be a renewal process & the interarrival times are called recurrence times (these may be ∞ which is a problem if recurrence is not certain). We may then identify u_t with

$$P(X_t = i \mid X_0 = i)$$

and the renewal theorem yields

$$P(X_t = i \mid X_0 = i) \to \frac{1}{\mu},$$

where μ = mean recurrence time.

eg Toss a coin with $p = P(H)$, $q = P(T)$. If H take 1 step to the right. If T take 1 step to the left. Start at 0 & step size = one. Let S_n = position after n steps ($S_0 = 0$ by convention). This is a simple random walk on the integers and $\{S_t, \underset{\substack{\uparrow \\ \text{integer}}}{t} \geq 0\}$ is clearly Markov (& time homogeneous).

Notice that the state space of $\{S_t\}$ is the integers & hence is countable. So we are dealing with a discrete time, time homogeneous Markov Chain. We are interested in the renewal events $S_t = 0$ & in particular the dist'n of the recurrence time.

We have

$$u_t = P(\text{renewal at time } t)$$
$$= 0 \quad , \text{if } t \text{ is odd}$$
$$= \binom{t}{t/2} (pq)^{t/2} \quad , t \text{ even}$$

Hence

$$U(z) = \sum_{k=0}^{\infty} \binom{2k}{k} (pq)^k z^{2k}$$
$$= (1 - 4pq z^2)^{-\frac{1}{2}}$$

$$\Rightarrow \quad G(z) = 1 - (1 - 4pq z^2)^{1/2}$$

Let T = recurrence time of state 0 (that is, starting in 0 it is the time to return to 0). We then have

$$G_T(z) = \frac{U(z) - 1}{U(z)}$$

$$= 1 - (1 - 4pqz^2)^{1/2}$$

& so

$$P(T=t) = \begin{cases} 0 & , t \text{ odd or } 0 \\ \binom{t}{t/2} \dfrac{(pq)^{t/2}}{t-1} & , t \text{ even} \end{cases}$$

[By convention $P(T=0) = 0$.]

<u>Note</u> Dist'n is restricted to even times (multiples of 2) & hence is not aperiodic (multiples of 1)

$$G_T(1) = 1 - |p-q|$$

& this is the probability that $T < \infty$.
If $p=q$ then recurrence is <u>certain</u>
(but note $G_T'(1) = \infty$ & so mean recurrence is ∞)

If $p \neq q$ then recurrence is <u>uncertain</u>. We call such states <u>transient</u>. States with finite recurrence times are called <u>recurrent</u> and recurrent states with finite mean recurrence times are called <u>positive recurrent</u> (ow they are <u>null recurrent</u>). States that can only recur at multiples of d ($d > 1$) are periodic – otherwise
integer
they are aperiodic.

<u>Markov Chains – states</u>

recurrent $<$ positive / null

transient – not certain to return

periodic – returns at multiples of $d > 1$
aperiodic – " " " " 1

(Non)homogeneous Poisson Process

$\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with rate / intensity function $\lambda(t)$ if

(i) $N(0)=0$ (ii) ind increments (iii) $P(N((t,t+h]) \geq 2) = o(h)$

(iv) $P(N((t,t+h])=1) = \lambda(t)\,h + o(h)$

Set $m(t) = \int_0^t \lambda(s)\,ds$ &

$$p_n(s) = P(N((t,t+s]) = n)$$

Then

$$p_0(s+h) = P(N((t,t+s])=0)\,P(N((t+s,t+s+h])=0)$$
$$= p_0(s)\,[1-\lambda(t+s)h + o(h)]$$

$$\Rightarrow \frac{p_0(s+h)-p_0(s)}{h} = -\lambda(t+s)\,p_0(s) + \frac{o(h)}{h}$$

$$\Rightarrow p_0'(s) = -\lambda(t+s)\,p_0(s)$$

$$\Rightarrow \log(p_0(s)) = -\int_0^t \lambda(t+u)\,du$$

$$\Rightarrow p_0(s) = e^{-[m(t+s)-m(t)]}$$

For $n \geq 1$

$$p_n(s+h) = P\{N((t,t+s+h]) = n\}$$

$$= P\{N((t,t+s]) = n,\ N((t+s,t+s+h]) = 0\}$$
$$+ P\{N((t,t+s]) = n-1,\ N((t+s,t+s+h]) = 1\}$$
$$+ P\{N((t,t+s]) < n-1,\ N((t+s,t+s+h]) \geq 2\}$$

$$= p_n(s)\, P\{N((t+s,t+s+h]) = 0\}$$
$$+ p_{n-1}(s)\, P\{N((t+s,t+s+h]) = 1\}$$
$$+ o(h)$$

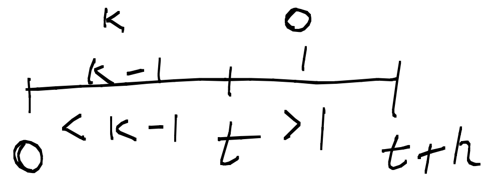
$$\Rightarrow p_n(s+h) = p_n(s)\left[1 - \lambda(t+s)h + o(h)\right]$$
$$+ p_{n-1}(s)\left[\lambda(t+s)h + o(h)\right] + o(h)$$

$$\Rightarrow p_n(s+h) = p_n(s)\left[1 - \lambda(t+s)h\right] + p_{n-1}(s)\lambda(t+s)h + o(h)$$

$$\Rightarrow \frac{p_n(s+h) - p_n(s)}{h} = -\lambda(t+s)p_n(s) + p_{n-1}(s)\lambda(t+s) + \frac{o(h)}{h}$$

$$\Rightarrow p_n'(s) = -\lambda(t+s)\, p_n(s) + p_{n-1}(s)\lambda(t+s) \qquad (*)$$

To show $N(t) \sim \text{Poisson}(\lambda t)$ set $p_k(t) = P(N(t) = k)$.



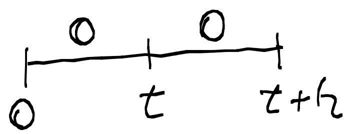
For $k > 0$ we have

$$p_k(t+h) = p_k(t)\, p_0(h) + p_{k-1}(t)\, p_1(h) + o(h)$$

$$\Rightarrow \quad \frac{p_k(t+h) - p_k(t)}{h} = p_k(t)\, \frac{p_0(h) - p_0(0)}{h} + p_{k-1}(t)\, \frac{p_1(h)}{h} + \frac{o(h)}{h}$$

$$\overset{h \downarrow 0}{\Rightarrow} \quad p_k'(t) = -\lambda\, p_k(t) + \lambda\, p_{k-1}(t) \qquad (*)$$

Note $p_0(0) = 1$, $\quad \dfrac{p_1(h)}{h} = \dfrac{1 - p_0(h) - o(h)}{h} = \dfrac{1 - p_0(h)}{h} + \dfrac{o(h)}{h}$

For $k = 0$ the picture is  which yields

$$p_0(t+h) = p_0(t)\, p_0(h)$$

which leads again to $(*)$ as $p_{-1}(t) = 0$.

From $(*)$ we see

$$\sum_{k=0}^{\infty} p_k'(t)\, z^k = -\lambda \sum_{k=0}^{\infty} p_k(t)\, z^k + \lambda \sum_{k=0}^{\infty} p_{k-1}(t)\, z^k$$

Notice

$$\sum_{k=0}^{\infty} p_{k-1}(t)\, z^k = z \sum_{k=1}^{\infty} p_{k-1}(t)\, z^{k-1}$$

$$= z \sum_{k=0}^{\infty} p_k(t)\, z^k$$

We then see by interchanging $\frac{d}{dt}$ with $\sum_{k=0}^{\infty}$ that

$$\frac{\partial G}{\partial t} = -\lambda G + \lambda z G ,$$

where $G(z,t) = E[z^{N(t)}]$ is the pgf of G.

$$\therefore \quad \frac{\partial G/\partial t}{G} = \lambda(z-1)$$

$$\Rightarrow \frac{\partial \log G}{\partial t} = \lambda(z-1) \Rightarrow \log(G) = \lambda(z-1)t + c .$$

Since $G(1,t)=1$ we have $c=0$ & so

$$G(z,t) = e^{\lambda t(z-1)}$$

$$\Rightarrow N(t) \sim \text{Poisson}(\lambda t)$$

We want to show

$$p_n(s) = e^{-[m(t+s)-m(t)]} \frac{[m(t+s)-m(t)]^n}{n!}$$

satisfies * for $n = 1, 2, \ldots$

For $n=1$

$$p_1(s) = e^{-[\quad]} [\quad]$$

$$\Rightarrow p_1'(s) = e^{-[\quad]} \lambda(t+s) - e^{-[\quad]} [\quad] \lambda(t+s)$$

& $p_0(s) = e^{-[\quad]}$

$\therefore$ (*) clearly holds for $n=1$

Now assume it true up to $n = N-1$ & consider the case $n = N$. This

$$p_N'(s) = -\lambda(t+s)\, p_N(s) + \lambda(t+s)\, p_{N-1}(s)$$

& $p_N(s) = e^{-[\quad]} \dfrac{[\quad]^N}{N!}$, where $[\quad] = m(t+s) - m(t)$

Now

$$p_{N-1}(\Delta) = e^{-[\ \ ]}\,\frac{[\ \ ]^{N-1}}{(N-1)!}$$

(by the induction hypothesis)

$$p_N'(\Delta) = \underbrace{e^{-[\ \ ]}\,\frac{[\ \ ]^{N-1}}{(N-1)!}\,\lambda(t+\Delta)}_{\lambda(t+\Delta)\,p_{N-1}(\Delta)} - \lambda(t+\Delta)\,\underbrace{e^{-[\ \ ]}\,\frac{[\ \ ]^{N}}{N!}}_{p_N(\Delta)}$$

$$p_N(\Delta) = e^{-[\ \ ]}\,[\ \ ]^N/N!$$

so that (*) holds for $n = N$ &

∴ for $n = 1, 2, \ldots$ by induction.

∴ $N((t, t+\Delta]) \sim \text{Poisson}([\ \ ])$

Markov Processes in time

Let T be a set of times (+ hence ordered). The stochastic process $\{X_t, t \in T\}$
↑ index set

is Markov if $\forall\ t_0 < t_1 < \cdots < t_n < t_{n+1}$

$$X_{t_{n+1}} | X_{t_n}, X_{t_{n-1}}, \ldots, X_{t_0}$$

$$\text{"="}\ X_{t_{n+1}} | X_{t_n}$$

The process has independent increments if the $X_{t_{i+1}} - X_{t_i}$ are independent. It has stationary increments if the dist'n of $X_{t+h} - X_t$ does not change with t!

A Markov process is time homogeneous if the conditional dist'ns $X_{t+h} | X_t$ do not change with t.
A Markov process is called a Markov Chain if the state space is countable.

The Poisson process

Consider $\{N_t, t \geq 0\}$ where $N_0 = 0$ and $N_t = \#$ of points in $[0, t]$ {there is an underlying point process}. Let $\lambda(t) \geq 0$ & set $m(t) = \int_0^t \lambda(u)\,du$. We say that $\{N_t, t \geq 0\}$ is a nonhomogeneous Poisson (counting) process of rate $\lambda(t)$ if it has independent increments &

$$N_t \sim \text{Poisson}(m(t))$$

If $\lambda(t) = \lambda$ then the process is homogeneous & is simply termed a Poisson process.

Notes

1. A nonhomogeneous Poisson process is a Markov Chain. If $\lambda(t) = \lambda$ then it is also time homogeneous.

2. $N((t_1, t_2)) = \#$ of points in (t_1, t_2)

$$\sim \text{Poisson}\Big(\underbrace{\int_{t_1}^{t_2} \lambda(u)\,du}_{m(t_2) - m(t_1)}\Big)$$

3. If $m(t)$ is strictly increasing (can be relaxed) then $\{N_{m^{-1}(t)}, t \geq 0\}$ is a homogeneous Poisson process of rate 1. To see this we note that the process has independent increments and

$$N_{m^{-1}(t)} \sim \text{Poisson}(\underbrace{m(m^{-1}(t))}_{t})$$

This tells us that one may view a nonhomogeneous process as a homogeneous one by donning appropriately warped glasses! Basically we redefine time.

4. $P(N(t, t+\Delta t)) = 1) = \lambda(t)\Delta t + o(\Delta t)$
$P(N(t, t+\Delta t)) > 1) = o(\Delta t)$
which, along with $N_0 = 0$ & ind increments, specify the nonhomogeneous Poisson process

The Galton Watson branching process

We start with 1 individual (can be generalized to >1) at generation/time 0. The individual has offspring according to an offspring dist'n with pgf $G(z)$. These offspring live for 1 generation + each of them independently has offspring according to G. This generates a Markov Chain $\{X_t, t=0,1,\cdots\}$ where X_t = # in the t th generation. Since $X_0=1$, X_1 has pgf $G(z) = E(z^{X_1})$. Set $\mu = E(X_1)$ = offspring mean. We are interested in calculating

$$\rho = \lim_{t\to\infty} \underbrace{P(X_t=0)}_{\text{increases with } t}$$

which we may term the probability of ultimate extinction.

<u>Theorem</u> Assume $0 < P(X_1 = 0) < 1$. Then

$$\mu < 1 \Rightarrow \rho = 1 \quad \text{(subcritical case)}$$
$$\mu = 1 \Rightarrow \rho = 1 \quad \text{(critical case)}$$
$$\mu > 1 \Rightarrow \rho < 1 \quad \text{(supercritical case)}$$

The proof is given in the text. To understand it we first need to obtain the dist'n of X_t. This is usually done via pgf's. So denote the pgf of X_t by G_t. We have

$$G_t(z) = E(z^{X_t}) = E[E(z^{X_t} | X_{t-1})]$$
$$= E(G(z)^{X_{t-1}})$$
$$= G_{t-1}[G(z)]$$

Since $G_0(z) = z$ we see

$$G_t(z) = \underbrace{G(G(\cdots(G}_{t \text{ of these}}(z))\cdots))$$
$$= t \text{ th iterate of } G = G_{(t)}(z)$$

For instance

$$G_2(z) = G(G(z)), \quad G_3(z) = G(G(G(z)))$$

Now $P(X_t = 0) = G_t(0) = G_{(t)}(0) = G(G_{(t-1)}(0))$

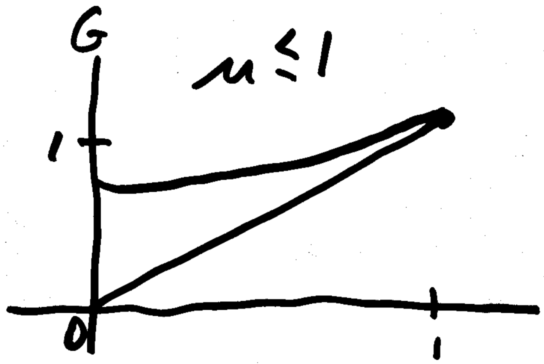
$$= G(P(X_{t-1} = 0))$$

$$\Rightarrow \lim_{t\to\infty} P(X_t = 0) = G\left(\lim_{t\to\infty} P(X_{t-1} = 0)\right) \quad \text{- since } G \text{ is cts}$$

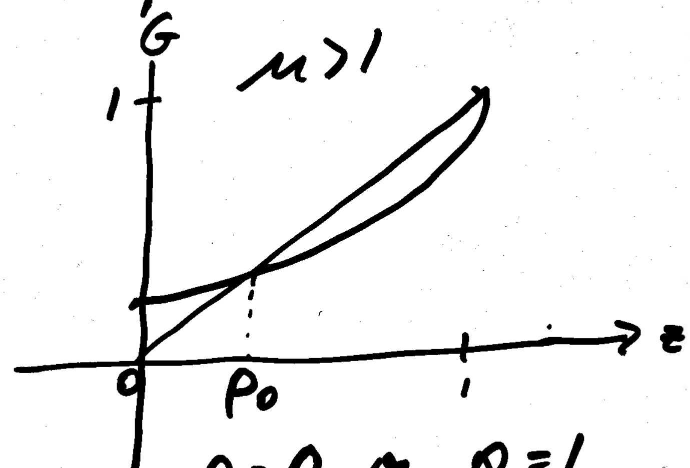
$$\Rightarrow \quad \rho = G(\rho) \qquad (*)$$

We need to solve (*) for $0 < \rho < 1$.

We have $G(z)$ is convex on $[0,1]$, $0 < \underbrace{G(0)}_{P(X_1=0)} < 1$ and $G'(1) = \mu$. So there are only 2 possible cases.



$\rho = 1$



$\rho = \rho_0$ or $\rho = 1$

Problem: $\rho = \rho_0$

Time Homogeneous Discrete Time Markov Chains

Consider $\{X_t, t = 0, 1, \cdots\}$. Set $\underset{\sim}{p}^{(t)}$ to be the pf of X_t. $\underset{\sim}{p}^{(0)}$ represents the initial dist'n. Define the n-step transition probabilities via

$$p_{ij}(n) = P(X_n = j \mid X_0 = i)$$

Set $P(n) = \underbrace{\{p_{ij}(n)\}_{i,j}}_{\text{matrix}}$ and call

$P = P(1)$ the transition matrix. Then

$$\underset{\sim}{p}^{(t)\prime} = \underset{\sim}{p}^{(0)\prime} P(t) = \underset{\sim}{p}^{(t-1)\prime} P$$

(this is just a simple consequence of the meaning of conditional probabilities)

If $\underset{\sim}{p}^{(t)} \to \underset{\sim}{\pi}$ ← limiting pf then we must have

$$\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P$$

Such a $\underset{\sim}{\pi}$ is called a stationary or equilibrium dist'n. In such a case $\underset{\sim}{\Pi}$ will not depend on the initial dist'n and the process is then said to be <u>ergodic</u>. From

$$\underset{\sim}{p}(t)' = \underset{\sim}{p}(0)' P(t)$$

we can then obtain $\lim_{t\to\infty} P(t)$ by taking $\underset{\sim}{p}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$, $\underset{\sim}{p}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$, etc...

and this yields

$$\lim_{t\to\infty} P(t) = \begin{pmatrix} \underset{\sim}{\Pi}' \\ \underset{\sim}{\Pi}' \\ \vdots \end{pmatrix}$$

Consider $t_1 < t_2 < t_3$. We then have

$$P(X_{t_3} = j \mid X_{t_1} = i) = \frac{P(X_{t_3} = j, X_{t_1} = i)}{P(X_{t_1} = i)}$$

$$= \sum_k \frac{P(X_{t_3}=j, X_{t_2}=k, X_{t_1}=i)}{P(X_{t_1}=i)}$$

$$= \sum_k P(X_{t_3}=j \mid X_{t_2}=k, X_{t_1}=i) \frac{P(X_{t_2}=k, X_{t_1}=i)}{P(X_{t_1}=i)}$$

$$= \sum_k P(X_{t_3}=j \mid X_{t_2}=k) P(X_{t_2}=k \mid X_{t_1}=i)$$

The equations

$$P(X_{t_3}=j \mid X_{t_1}=i) = \sum_k P(X_{t_2}=k \mid X_{t_1}=i) P(X_{t_3}=j \mid X_{t_2}=k)$$

are called the Chapman Kolmogorov equations (CKE). In the time homogeneous case they reduce to

$$P(s+t) = P(s)\, P(t)$$

(in matrix form)

Since $P(0) = I$ we conclude

$$P(t) = P^t$$

Hence

$$\underset{\sim}{p}(t) = \underset{\sim}{p}(0) P^t$$

When does $\underset{\sim}{p}(t) \to \underset{\sim}{\pi}$?

Not simple (see STA 447), but we almost have an answer via the renewal theorem as the point process which follows state i is a renewal process. As a consequence the recurrence times (+ mean recurrence times) are key.

<u>Def'n</u> State i is recurrent if starting from i one is certain to return.

Notice that this is simply stating that the recurrence time is $< \infty$ wp1.

Of course if the recurrence time is finite then the # of recurrences must be ∞ (ie the number of renewals). If $P(\text{recurrence time} = \infty) > 0$ then there is a > 0 probability of never returning to i (either you do or you don't) so that the number of returns would be geometric & the mean # of returns would be finite. Now starting from i at $t=0$ the # of returns is

$$\sum_{t=1}^{\infty} I(X_t = i)$$

with mean

$$\sum_{t=1}^{\infty} E(I(X_t = i) | X_0 = i) = \sum_{t=1}^{\infty} P_{ii}^{(t)}$$

It follows that state i is recurrent iff

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$$

<u>Def'n</u> A recurrent state is <u>positive recurrent</u> if the mean return time is finite. Otherwise it is null recurrent.

<u>Remark</u> In the simple symmetric random walk on the integers we were able to determine the type of recurrent state (for the origin)

<u>Def'n</u> A state which is not recurrent is called <u>transient</u>.

<u>Def'n</u> A state which can only return at multiples of $d > 1$ is called <u>periodic</u>. The smallest such d is the period.

<u>Def'n</u> j is accessible from i $(i \to j)$ if $P_{ij}^{(n)} > 0$ for some $n \geq 0$.

<u>Def'n</u> i & j communicate if $i \to j$ and $j \to i$. We write $i \leftrightarrow j$.

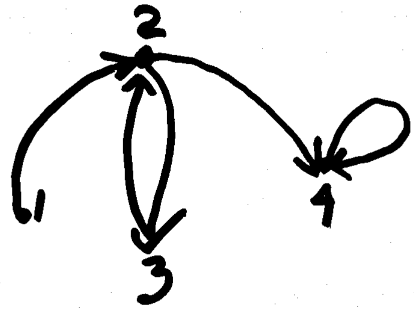
<u>Def'n</u> A Markov Chain is irreducible if all states communicate.

<u>Theorem</u> $i \leftrightarrow j \Rightarrow$ they are the same type (recurrent, transient, etc ...)

<u>Remark</u>

1. States which are not periodic are termed aperiodic. The behaviour of a periodic state can be determined by looking at times kd, $k = 0, 1, \ldots$.
2. For a finite Markov Chain which is irreducible all states must be <u>positive recurrent</u>.

3. A sketch of the one-step transition probabilities is often helpful.



Here $1 \to 3$, $2 \leftrightarrow 3$, $1 \not\leftrightarrow 2$ & $1 \to 4$

Notice that once in 4 there is nowhere to go. It is an <u>absorbing state</u>.

<u>Theorem</u> In an irreducible, aperiodic, positive recurrent Markov Chain

$$\lim_{t \to \infty} P(X_t = i \mid X_0 = i) = \lim_{t \to \infty} P(X_t = i) = \pi_i$$

where $\pi_i = 1/\text{mean recurrence time}$, $\underset{\sim}{\pi}$ is a pf & $\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P$.

eg Consider a two state Markov Chain with

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

where $0 < \alpha, \beta < 1$. Here all states communicate & are aperiodic. Since the chain is finite they are all positive recurrent. Hence

$$\lim_{t \to \infty} \underset{\sim}{p}(t) = \underset{\sim}{\pi}$$

where $\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P$. The equation

$$\boxed{\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P}$$

can be solved for π_1 & π_2 yielding

$$\pi_1 = \beta/(\alpha+\beta) \qquad \pi_2 = \alpha/(\alpha+\beta)$$

$$\therefore \; P^t \to \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}$$

see p159 for a calculation of P^t !!