Week #4

binomial

iid Bernoulli trials yielding

$$\underset{2\times1}{\underset{\sim}{X}_1}, \ \underset{2\times1}{\underset{\sim}{X}_2}, \ \ldots, \ \underset{2\times1}{\underset{\sim}{X}_N} \longrightarrow \text{iid } \underset{\sim}{X}$$

$p_1 + p_2 = 1$

Note possible values of $\underset{\sim}{X}$ are $\overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{p_2}$ or $\overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{p_1}$

jt pgf of the components of $\underset{\sim}{X}$ or just the the pgf of $\underset{\sim}{X}$

$$G_{\underset{\sim}{X}}(\underset{\sim}{z}) = E\left(\underset{\sim}{z}^{\underset{\sim}{X}}\right) = E\left(z_1^{\text{1st comp of } \underset{\sim}{X}} \, z_2^{\text{2nd comp of } \underset{\sim}{X}}\right)$$

$$= z_2 p_2 + z_1 p_1$$

$$= p_1 z_1 + p_2 z_2$$

= pgf of a "vector Bernoulli"

Now set

$$\underset{\sim}{Y} = \underset{\sim}{X}_1 + \cdots + \underset{\sim}{X}_N$$

$$\Rightarrow \quad G_{\underset{\sim}{Y}}(\underset{\sim}{z}) = \left(p_1 z_1 + p_2 z_2\right)^N = \sum_{y_1+y_2=N} \binom{N}{y_1, y_2} p_1^{y_1} p_2^{y_2} z_1^{y_1} z_2^{y_2}$$

$$\Rightarrow \quad P(\underset{\sim}{Y} = \underset{\sim}{y}) = \binom{N}{y_1, y_2} p_1^{y_1} p_2^{y_2}$$

Note $z_2 = 1 \Rightarrow (p_1 z_1 + p_2)^N$ is the pgf of the 1st component of $\underset{\sim}{X}$ (this is the binomial(N, p_1) as you knew it).

Extend to the multinomial

$$\underset{M\times 1}{\underset{\sim}{X}_1}, \underset{M\times 1}{\underset{\sim}{X}_2}, \dots, \underset{M\times 1}{\underset{\sim}{X}_N} \quad \text{iid } \underset{\sim}{X}$$

$\underset{\sim}{X}$ has one component $= 1$ + the rest 0. The probability that the 1 is in the ith place we call p_i $(i = 1, \dots, M)$. The pgf of $\underset{\sim}{X}$ is

$$(p_1 z_1 + p_2 z_2 + \cdots + p_M z_M)$$

& letting

$$\underset{\sim}{Y} = \underset{\sim}{X}_1 + \cdots + \underset{\sim}{X}_N$$

we get

$$G_{\underset{\sim}{Y}}(\underset{\sim}{z}) = (p_1 z_1 + \cdots + p_M z_M)^N$$

$$\Rightarrow \quad P(Y = y) = \underbrace{\binom{N}{y}}_{\binom{N}{y_1, \dots, y_M}} p_1^{y_1} \cdots p_M^{y_M}$$

$\underline{\text{Note}}$ $\frac{\partial^2}{\partial z_1 \partial z_2} G(\underset{\sim}{z}) = E\left(Y_1 z_1^{Y_1 - 1} Y_2 z_2^{Y_2 - 1} z_3^{Y_3} \cdots\right)$ works for

$\overset{\underset{\sim}{z} = \underset{\sim}{1}}{=} E(Y_1 Y_2)$ all counting rvectors

etc. . . .

Can use to get the $\mathrm{cov}(Y_i, Y_j)$ for counting rvec's $\underset{\sim}{Y}$.

Properties of covariances

$(\mathrm{cov}(X, Y) = E(XY) - E(X)E(Y))$

verify

$$\mathrm{cov}(X, X) = \mathrm{Var}(X)$$

$$\mathrm{cov}(X + c, Y + d) = \mathrm{cov}(X, Y)$$

$$\mathrm{cov}(aX, bY) = ab\,\mathrm{cov}(X, Y)$$

$$\mathrm{cov}\left(\sum_i X_i, \sum_j Y_j\right) = \sum_{i,j} \mathrm{cov}(X_i, Y_j)$$

eg $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$,

$X_3 \sim \text{Poisson}(\lambda_3)$

$$\left.\begin{aligned} U &= X_1 + X_2 \\ V &= X_2 + X_3 \end{aligned}\right\} \text{ easy to get the pgf}$$

$$= E\left(z_1^U z_2^V\right)$$

$$= E\left(z_1^{X_1 + X_2} z_2^{X_2 + X_3}\right)$$

$$= E\left(z_1^{X_1} (z_1 z_2)^{X_2} z_2^{X_3}\right)$$

$$= E\left(z_1^{X_1}\right) E(z_1 z_2)^{X_2} E\left(z_2^{X_3}\right)$$

$$= e^{\lambda_1(z_1-1)} e^{\lambda_2(z_1 z_2-1)} e^{\lambda_3(z_2-1)}$$

– Use this to get $E(UV)$ – <u>imp</u>
& then get $\mathrm{cov}(U, V)$ } 15 minutes

<u>Another way</u> $\mathrm{cov}(U, V) = \mathrm{cov}(X_1 + X_2, X_2 + X_3)$

$$= \mathrm{cov}(X_1, X_2) + \mathrm{cov}(X_1, X_3) + \mathrm{cov}(X_2, X_2) + \mathrm{cov}(X_2, X_3)$$

$$= \mathrm{cov}(X_2, X_2) = \mathrm{Var}(X_2) = \lambda_2$$

<u>Note</u> X, Y ind $\Rightarrow E(XY) = E(X)E(Y) \Rightarrow \mathrm{cov}(X, Y) = 0$

<u>eg</u> $Z \sim N(0,1)$. Let $X = Z$, $Y = Z^2$. Then
X & Y are <u>dependent</u> but

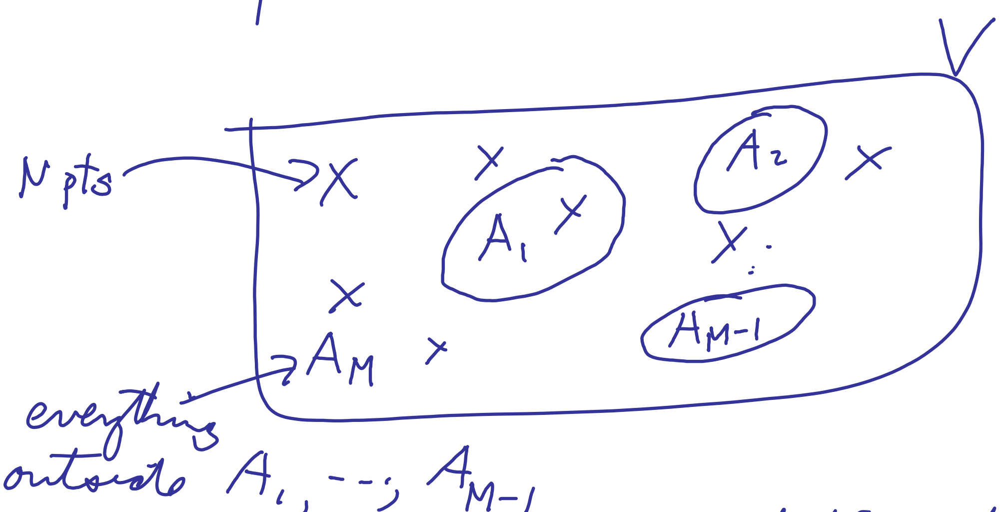
$$E(XY) = E(Z Z^2) = E(Z^3) = 0 = \underbrace{E(X)}_{0} \underbrace{E(Y)}_{1}$$

& so X & Y are uncorrelated.

Spatial Poisson process

Background A rvec is uniform on a set if its pf or pdf is constant there.

Throw N points onto V in a uniform way



$\{A_1, \ldots, A_{M-1}, A_M\}$ is a partition of V.

Let $N(A) = \#$ of pts in a set $A \subset V$.

$$\sim \text{binomial}(N, |A|/|V|)$$

Also $N(A_1), \ldots, N(A_M)$ is our multinomial with $p_i = |A_i|/|V|$

+ the pgf is

$$(p_1 z_1 + \cdots + p_{M-1} z_{M-1} + p_M z_M)^N$$

$\Rightarrow$ pgf of $N(A_1), \cdots, N(A_{M-1})$

is $\left(p_1 z_1 + \cdots + p_{M-1} z_{M-1} + p_M\right)^N$

$p_M = 1 - (p_1 + \cdots + p_{M-1})$

Let V "$\to$" $\mathbb{R}^{(\;)}$ and $N \to \infty$ $\Big\}$ keep $\frac{N}{|V|} = \rho$ fixed

ρ called λ eventually

ρ is like a density of pts

In the limit you get a Poisson point process on $\mathbb{R}^{(\;)}$ of "rate" ρ.

Poisson <u>spatial</u> process, Poisson counting process, Poisson process.

<u>Why is it called Poisson?</u>

The jt pgf of $N(A_1), \cdots, N(A_{M-1})$ is

$$(*) \quad \left[p_1(z_1-1) + \cdots + p_{M-1}(z_{M-1}-1) + 1\right]^N$$

& $p_i = \dfrac{|A_i|}{|V|} = \dfrac{|A_i|\,N}{|V|\,N} = \dfrac{\rho\,|A_i|}{N}$

so that $(*)$ is

$$\left[1 + \frac{\rho|A_1|(z_1-1) + \rho|A_2|(z_2-1) + \cdots + \rho|A_{M-1}|(z_{M-1}-1)}{N}\right]^N$$

$$\xrightarrow{N\to\infty} e^{\rho|A_1|(z_1-1) + \cdots + \rho|A_{M-1}|(z_{M-1}-1)}$$

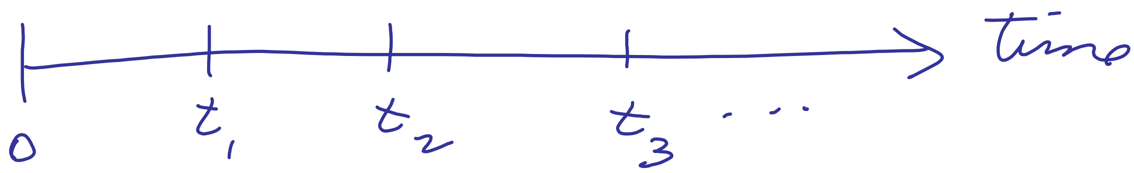
= pgf of $M-1$ independent Poisson's with means $\rho|A_1|, \ldots, \rho|A_{M-1}|$

<u>Note</u> For a Poisson process the #'s in disjoint regions are independent Poisson's & the # in a set A is Poisson with mean $\underline{\underline{\text{rate}}} \times |A|$

ρ or λ

usual

<u>Special case</u> Poisson process of rate λ on $t \geq 0$.



If $t_0 < t_1 < t_2 < \cdots$ & $N(t) =$ # of points in $[0, t]$

then $N(t_i) - N(t_{i-1}) =$ # of pts in $(t_{i-1}, t_i]$, $i = 1, 2, \ldots$

$\{N(t) : t \geq 0\}$ is a Poisson counting process of rate λ on $t \geq 0$.

Note that the $N(t_i) - N(t_{i-1})$ are independent (independent increments).

$\left.\begin{array}{l}\text{Times between pts are rv's} \\ \text{Time to the } n\text{th pt is a rv}\end{array}\right\}$ want to study these

Let S_r = time to the rth pt &

$X_1, X_2, \ldots$ be the times between pts

X_2 ← time from 1st pt to 2nd

X_1 ← time from 0 to the 1st pt

<u>dist'n of X_1</u>

Let $F(x) = P(X_1 \le x)$ — df of X_1

$\bar{F}(x) = P(X_1 > x)$ — Tail probability f'n of X_1

Assume F (or $\bar{F}$) determine the dist'n (which it does but hard to prove).

Here

$$F(x) = 0 \text{ unless } x > 0$$

Let $x > 0$. Then

$$\bar{F}(x) = P(X_1 > x) = P(N(x) = 0)$$

But

$$N(x) \sim \text{Poisson}(\lambda x)$$

$$\Rightarrow P(N(x) = 0) = e^{-\lambda x}$$

$$\therefore \quad \bar{F}(x) = e^{-\lambda x}, \quad x > 0$$
$$= 1, \quad ow$$

Notice

$$\bar{F}'(x) = -\lambda e^{-\lambda x}, \quad x > 0$$

&

$$F'(x) = \lambda e^{-\lambda x}, \quad x > 0$$

which is the exponential (λ) pdf. So

$$X_1 \sim \text{exponential}(\lambda)$$

<u>Note</u> If a pdf exists then $\bar{F}'(x) = -f(x)$.
The exponential(λ) dist'n, by def'n, has pdf

$$\left.\begin{aligned} f(x) &= \lambda e^{-\lambda x}, & x > 0 \\ &= 0, & ow \end{aligned}\right\} \text{mean} = \frac{1}{\lambda}$$

If $F'(x)$ is a pdf then we are dealing with an absolutely cts dist'n.

<u>For S_r</u> = time to rth point

$$\bar{F}_r(x) = P(S_r > x)$$

Let $x > 0$ then

$$\bar{F}_r(x) = P(S_r > x) = P(N(x) \le r-1)$$

$$= \sum_{k=0}^{r-1} e^{-\lambda x} \frac{(\lambda x)^k}{k!}$$

Now calculate $\bar{F}_r'(x)$ to get - the pdf of a gamma (check this).

<u>Extra material (basic conditioning, a review)</u>

<u>Def'n</u> If $P(A) > 0$ define $E(Y|A) = \dfrac{E[Y I(A)]}{E[I(A)]}$.

It is easily seen that, for fixed A, $E(\cdot|A)$ satisfies the Axioms for E (verify this).

<u>Def'n</u> $P(B|A) = E(I(B)|A)$

Of course $P(B|A) = P(BA)/P(A)$ as before

If $\underset{\sim}{X}$ is discrete then we set

$$r(\underset{\sim}{x}) = E(Y \mid \underset{\sim}{X} = \underset{\sim}{x})$$

and

$$E(Y|X) = r(\underset{\sim}{X})$$

If Y is also discrete then it is easily verified

$$E(Y) = E[E(Y|\underset{\sim}{X})]$$

and

$$E(Y|\underset{\sim}{X}_1) = E[E(Y|\underset{\sim}{X}_1, \underset{\sim}{X}_2) \mid \underset{\sim}{X}_1]$$

A further property is

$$E[Y g(\underset{\sim}{X})] = E[E(Y|\underset{\sim}{X})\, g(\underset{\sim}{X})], \quad \text{"}\forall\text{" real } g$$

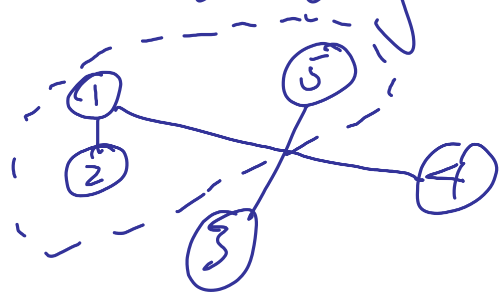
These last three properties are true in general and are easily verified when $\begin{pmatrix} Y \\ \underset{\sim}{X} \end{pmatrix}$ is cts with pdf $f(y, \underset{\sim}{x}')$.

Finally, it can be shown that $E(Y|\underset{\sim}{X})$ satisfies (see Th 5.3.1)

$$E(Y - E(Y|\underset{\sim}{X}))^2 = \min_{\text{real } h} E(Y - h(\underset{\sim}{X}))^2$$

Application of the uniform + Bernoulli / Indicator

N – towns } at most 1 road between
m – roads } any 2 towns



$\frac{2}{3}$ of roads go into to dotted collection

In fact there is always a collection of towns $\ni$ at least half of roads lead into it.

Sol'n Go to each town + toss a fair coin. If H put town into a collection S. S is a random collection of towns. Note that there are 2^N possible "values" that S can be.
Let X = # of roads leading into S from outside. We need to show there is

a possible value of $X \geq m/2$. Since X is a (positive) counting rv this will hold if $E(X) \geq m/2$. Now, let

$$I_j = I(\{\text{road } j \text{ leads into } S\})$$

Then

$$X = \sum_{j=1}^{m} I_j$$

&

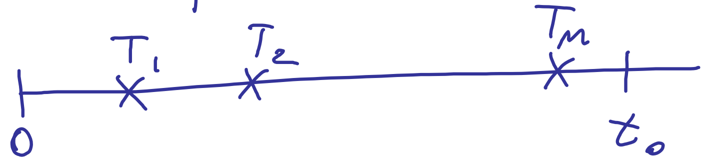
$$E(X) = m \, E(I_1) = m \, P(\text{road } 1 \text{ leads into } S)$$

$$= m \, P(\text{one H + one T for the 2 tosses})$$

$$= m/2$$

Let $\{N(t) : t \geq 0\}$ be a Poisson process of rate λ on $t \geq 0$.

Suppose you <u>know</u> $N(t_0) = m$. Call the times of the points $T_1 < T_2 < \cdots < T_m$



We want the pdf of T_n.

Approach

Use cdf F or $\bar{F} = 1 - F$

Aside

$$F(x) = P(X \le x), \quad \forall x \in \mathbb{R}$$

For a cts rv X

$$F(x) = \int_{-\infty}^{x} \overset{\text{pdf}}{f}(t)\,dt$$

$\Longrightarrow$ at continuity pts

$$F'(x) = f(x)$$

In fact we could take F' to be an equivalent pdf.

$$F(b) - F(a) = P(a < X \le b)$$

$$\left\{ \therefore \overbrace{P(X \le a)}^{F(a)} + P(a < X \le b) = \overbrace{P(X \le b)}^{F(b)} \right\}$$

$$= \int_a^b F'(t)\,dt \quad \boxed{\text{FLofC}}$$

<u>Remark</u> If F is cts it is possible for $F'(x) = 0$ for <u>almost every x</u> <u>Feller</u>
(places where not true have <u>length</u> 0)
& so in that case $\int_{-\infty}^{\infty} F'(x)\,dx = 0$.

$\bar{F}'(x) = -f(x)$. So knowing F or $\bar{F}$
$\Rightarrow$ know the dist'n.

By the way,
$x_1 < x_2 \Rightarrow P(X \le x_1) \le P(X \le x_2)$
$\therefore \{X \le x_1\} \Rightarrow \{X \le x_2\}$

$\therefore$ F is increasing.
If $x_n \downarrow a \Rightarrow \{X \le x_n\} \downarrow \{X \le a\}$
$$\Rightarrow P(X \le x_n) \to P(X \le a)$$
(continuity property of P)
$\therefore$ F is right cts

If $x_n \uparrow \infty \Rightarrow \{X \le x_n\} \uparrow \{X < \infty\}$

$\Rightarrow F(x_n) \to P(X<\infty) = 1$

$$\therefore \underbrace{x \to \infty \Rightarrow F(x) \to 1}_{F(\infty) = 1}$$

If $x_n \downarrow -\infty \Rightarrow \{X \le x_n\} \downarrow \emptyset$

$\Rightarrow F(x_n) \to 0$

$$\therefore \underbrace{x \to -\infty \Rightarrow F(x) \to 0}_{F(-\infty) = 0}$$

Note ① $x_n \uparrow a \Rightarrow \{X \le x_n\} \uparrow \{X < a\}$

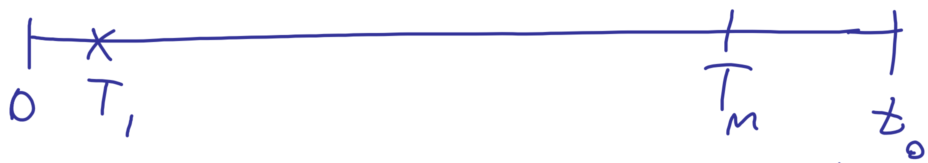
$$\therefore \lim_{x \uparrow a} F(x) = P(X<a) \ne F(a)$$

$$P(X=a) = F(a) - \lim_{x\uparrow a} F(x)$$

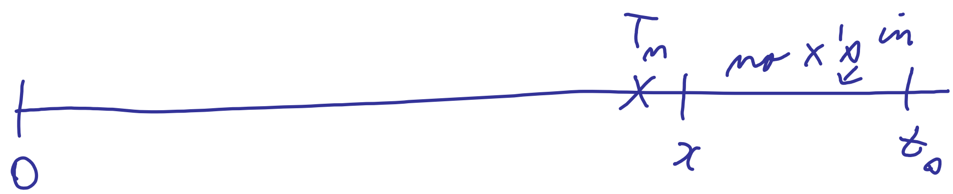
② Knowing $F \Rightarrow$ know $P(a \underset{(}{\le} X \underset{(}{\le}, b)$

not obvious $\Rightarrow$ know the dist'n

Back to our Poisson process.



$$F_{(m)}(x) = P(T_m \le x) \quad \left\{ = P\left(T_m \le x \,\middle|\, N(t_0) = m\right) \right\}$$



$$= P\left(N((x, t_0]) = 0 \,\middle|\, N(t_0) = m\right)$$

$$= \frac{P\left(N((x, t_0]) = 0,\ N(t_0) = m\right)}{P(N(t_0) = m)}$$

$$= \frac{P\left(N(x) = m,\ N((x, t_0]) = 0\right)}{P(N(t_0) = m)}$$

$$= \frac{P(N(x) = m)\ P\left(N((x, t_0]) = 0\right)}{P(N(t_0) = m)}$$

$$= \frac{P(N(x)=m)\; P(N((x, t_0]) = 0)}{P(N(t_0) = m)}$$

$$= \frac{e^{-\lambda x} \frac{(\lambda x)^m}{m!} \; e^{-\lambda(t_0 - x)}}{e^{-\lambda t_0} \frac{(\lambda t_0)^m}{m!}}$$

$$= \left(\frac{x}{t_0}\right)^m$$

$\therefore$ conditional on $N(t_0) = m$,

$$F_{(m)}(x) = \left(\frac{x}{t_0}\right)^m, \quad 0 \le x \le t_0$$

$$= 1, \quad x > t_0$$

$$= 0, \quad \text{ow}$$

& so

$$f_{(n)}(x) = \frac{n}{t_0}\left(\frac{x}{t_0}\right)^{n-1}, \quad 0 \le x \le t_0$$
$$= 0, \quad \text{ow}$$

If $t_0 = 1$ then
$$f_{(n)}(x) = n\, x^{n-1}, \quad 0 \le x \le 1$$
$$= 0, \quad \text{ow}$$

Remark

$$r(x) = E(\underset{\text{discrete}}{Y \mid X} = x) = \sum_y y\, f(y \mid x)$$

$$f(y \mid x) = \frac{f(x,y)}{f(x)}, \quad f(x,y) = P(X = x, Y = y), \quad f(x) = P(X = x)$$

$$\boxed{E(Y \mid X) = r(X)}$$

$$\boxed{E\left[E(Y \mid X)\right] = E(Y)}$$

$$E\left[E(z^Y \mid X)\right] = E(z^Y)$$

can be extended to the cts case & then

$$r(x) = E(Y \mid X = x) = \int_{-\infty}^{\infty} y\, \underbrace{f(y \mid x)}_{\text{cond pdf}}\, dy$$