Lecture 5

Apples into barrels

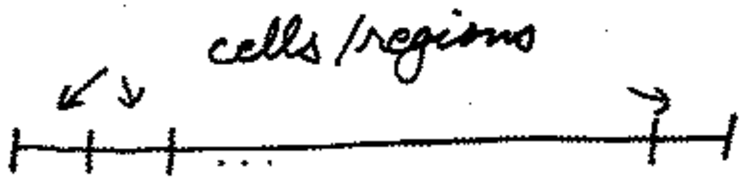
M barrels

N apples

2 cases: apples "identical"; all different

all different $\to M^N$

"identical" $\to$ fewer



N "points" into M cells

If done uniformly + independently then

$$P(Y_1 = k_1, \dots, Y_M = k_M) = \binom{N}{k_1, \dots, k_M} M^{-N}$$

which is the multinomial$(N; \frac{1}{M}, \dots, \frac{1}{M})$ or Maxwell-Boltzmann dist'n ("occupancy statistics")

<u>Remark</u>

(i) If the points are distinguishable then the # of arrangements of the pts in the cells is

$$\sum_{k_1, \dots, k_M} \binom{N}{k_1, \dots, k_M} = (1 + \cdots + 1)^N = M^N$$

Of course different arrangements may lead to the <u>same</u> counts.

(ii) $Y_1 \sim$ binomial$(N, \frac{1}{M}) \overset{d}{\approx}$ Poisson(μ)

if $\frac{N}{M} \approx \mu$ + N is large.

If the points are indistinguishable then the # of arrangements will be fewer. We can count them as follows:

|xxx|x|————————|xxxx|

N — x's
M-1 — | ← longer than the end lines

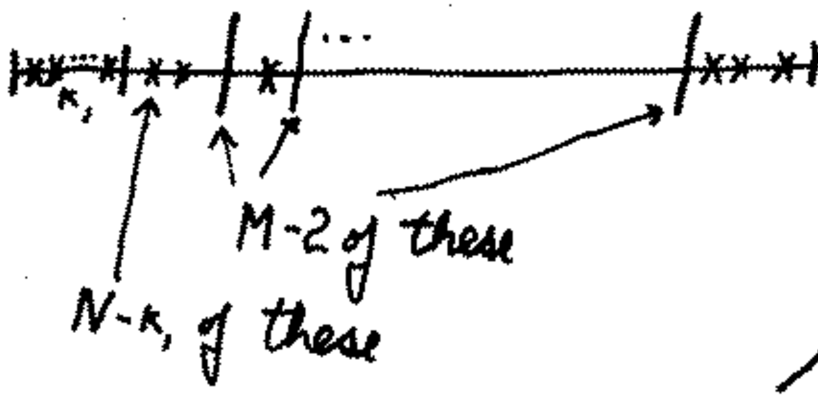
As we move the |'s + x's around we obtain the different ways of putting the N points into the M cells (we are only keeping track of the counts in each cell). This is identical to moving M-1 1's + N 0's around in N+M-1 cells + asking # of arrangements! So we obtain for the # of arrangements

$$\binom{N+M-1}{N} \text{ or } \binom{N+M-1}{M-1}$$

Notice that each arrangement corresponds to counts in each of the M cells. If equal probability is assigned to each then we have the "Bose-Einstein statistics"

In particular

$$P(Y_1 = k_1) = \binom{N-k_1+M-2}{M-2} \Big/ \binom{N+M-1}{M-1}$$



same reasoning as before

Of course

$$P(Y_1 = k_1) = \binom{N-k_1+M-2}{N-k_1} \Big/ \binom{N+M-1}{N}$$

& if $\frac{N}{M} \approx \lambda$ & $N \to \infty$ then

$$P(Y_1 = k_1) \approx \left(\frac{\lambda}{1+\lambda}\right)^{k_1}\left(\frac{1}{1+\lambda}\right) \quad , k_1 = 0, 1, \cdots$$

which is the geometric (starting at 0) distribution.

Extra material (basic conditioning, a review)

Def'n If $P(A) > 0$ define $E(Y|A) = \dfrac{E[Y\, I(A)]}{E[I(A)]}$.

It is easily seen that, for fixed A, $E(\cdot|A)$ satisfies the Axioms for E (verify this).

Def'n $P(B|A) = E(I(B)|A)$

Of course $P(B|A) = P(BA)/P(A)$ as before

If $\underset{\sim}{X}$ is discrete then we set

$$r(\underset{\sim}{x}) = E(Y \mid \underset{\sim}{X} = \underset{\sim}{x})$$

and

$$E(Y \mid X) = r(\underset{\sim}{X})$$

If Y is also discrete then it is easily verified

$$E(Y) = E[E(Y \mid \underset{\sim}{X})]$$

and

$$E(Y \mid \underset{\sim}{X}_1) = E[E(Y \mid \underset{\sim}{X}_1, \underset{\sim}{X}_2) \mid \underset{\sim}{X}_1]$$

A further property is

$$E[Y g(\underset{\sim}{X})] = E[E(Y \mid \underset{\sim}{X})\, g(\underset{\sim}{X})], \quad \text{"}\forall\text{" real } g$$

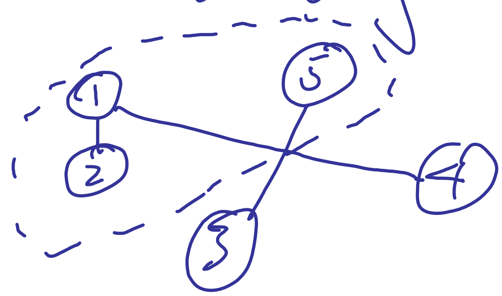
These last three properties are true in general and are easily verified when $\begin{pmatrix} Y \\ \underset{\sim}{X} \end{pmatrix}$ is cts with pdf $f(y, \underset{\sim}{x}')$.

Finally, it can be shown that $E(Y \mid \underset{\sim}{X})$ satisfies (see Th 5.3.1)

$$E(Y - E(Y \mid \underset{\sim}{X}))^2 = \min_{\text{real } h} E(Y - h(\underset{\sim}{X}))^2$$

Application of the uniform + Bernoulli / Indicator

N – towns } at most 1 road between
m – roads } any 2 towns



$\frac{2}{3}$ of roads go into to dotted collection

In fact there is always at collection of towns $\ni$ at least half of roads lead into it.

Sol'n Go to each town + toss a fair coin. If H put town into a collection S. S is a random collection of towns. Note that there are 2^N possible "values" that S can be.
Let X = # of roads leading into S from outside. We need to show there is

a possible value of $X \geq m/2$. Since X is a (positive) counting rv this will hold if $E(X) \geq m/2$. Now, let

$$I_j = I(\{\text{road } j \text{ leads into } S\})$$

Then

$$X = \sum_{j=1}^{m} I_j$$

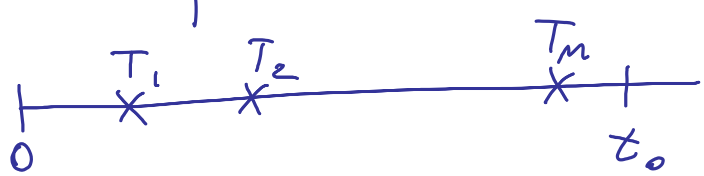
&

$$E(X) = m\, E(I_1) = m\, P(\text{road 1 leads into } S)$$

$$= m\, P(\text{one H \& one T for the 2 tosses})$$

$$= m/2$$

Let $\{N(t) : t \geq 0\}$ be a Poisson process of rate λ on $t \geq 0$. Suppose you __know__ $N(t_0) = n$. Call the times of the points $T_1 < T_2 < \cdots < T_n$



We want the pdf of T_n.

<u>Approach</u>
Use cdf F or $\bar{F} = 1 - F$

<u>Aside</u>

$$F(x) = P(X \leq x) \quad , \quad \forall x \in \mathbb{R}$$

For a cts rv X

$$F(x) = \int_{-\infty}^{x} \underbrace{f}_{\text{pdf}}(t)\,dt$$

$\Longrightarrow$ at continuity pts

$$F'(x) = f(x)$$

In fact we could take F' to be an equivalent pdf.

$$F(b) - F(a) = P(a < X \leq b)$$

$$\left\{ \overbrace{P(X \leq a)}^{F(a)} + P(a < X \leq b) = \overbrace{P(X \leq b)}^{F(b)} \right\}$$

$$= \int_a^b F'(t)\,dt \qquad \boxed{FLofC}$$

Remark If F is cts it is possible for $F'(x) = 0$ for almost every x (places where not true have length 0) & so in that case $\int_{-\infty}^{\infty} F'(x)\,dx = 0$.

Feller

$\bar{F}'(x) = -f(x)$. So knowing F or $\bar{F}$ $\Rightarrow$ know the dist'n.

By the way,

$$x_1 < x_2 \Rightarrow P(X \le x_1) \le P(X \le x_2)$$

$$\therefore \{X \le x_1\} \Rightarrow \{X \le x_2\}$$

$\therefore$ F is increasing.

If $x_n \downarrow a \Rightarrow \{X \le x_n\} \downarrow \{X \le a\}$

$$\Rightarrow P(X \le x_n) \to P(X \le a)$$

(continuity property of P)

$\therefore$ F is right cts

If $x_n \uparrow \infty \Rightarrow \{X \le x_n\} \uparrow \{X < \infty\}$

$\Rightarrow F(x_n) \to P(X < \infty) = 1$

$$\therefore \underbrace{x \to \infty \Rightarrow F(x) \to 1}_{F(\infty) = 1}$$

If $x_n \downarrow -\infty \Rightarrow \{X \le x_n\} \downarrow \emptyset$

$\Rightarrow F(x_n) \to 0$

$$\therefore \underbrace{x \to -\infty \Rightarrow F(x) \to 0}_{F(-\infty) = 0}$$

Note ① $x_n \uparrow a \Rightarrow \{X \le x_n\} \uparrow \{X < a\}$

$\vdots$

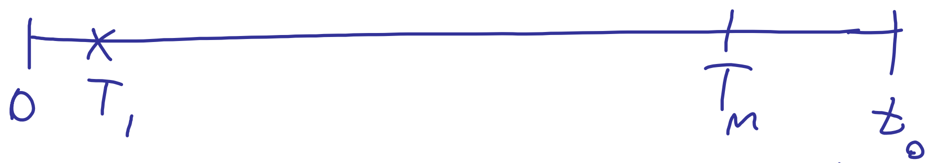
$$\therefore \lim_{x \uparrow a} F(x) = P(X < a) \ne F(a)$$

$$P(X = a) = F(a) - \lim_{x \uparrow a} F(x)$$

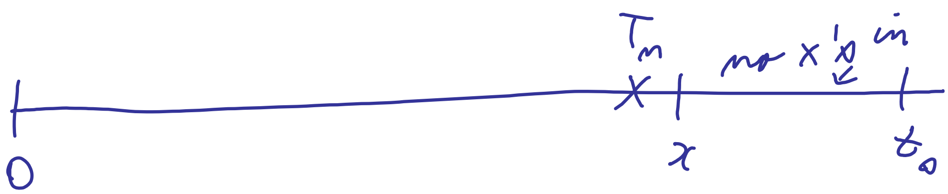
② Knowing $F \Rightarrow$ know $P(a \underset{(}{\le} X \underset{,}{(\le} b))$

$\overset{\text{not obvious}}{\Rightarrow}$ know the dist'n

Back to our Poisson process.



$$F_{(m)}(x) = P(T_m \le x) \quad \left\{ = P\left(T_m \le x \,\middle|\, N(t_0) = m\right) \right\}$$



$$= P\left(N((x, t_0]) = 0 \,\middle|\, N(t_0) = m\right)$$

$$= \frac{P\left(N((x, t_0]) = 0,\ N(t_0) = m\right)}{P(N(t_0) = m)}$$

$$= \frac{P\left(N(x) = m,\ N((x, t_0]) = 0\right)}{P(N(t_0) = m)}$$

$$= \frac{P(N(x) = m)\ P\left(N((x, t_0]) = 0\right)}{P(N(t_0) = m)}$$

$$= \frac{P(N(x)=m)\, P(N((x, t_0]) = 0)}{P(N(t_0) = m)}$$

$$= \frac{e^{-\lambda x}\frac{(\lambda x)^m}{m!}\; e^{-\lambda(t_0 - x)}}{e^{-\lambda t_0}\frac{(\lambda t_0)^m}{m!}}$$

$$= \left(\frac{x}{t_0}\right)^m$$

$\therefore$ conditional on $N(t_0) = m$,

$$F_{(m)}(x) = \left(\frac{x}{t_0}\right)^m \quad , \quad 0 \le x \le t_0$$

$$= 1 \quad , \quad x > t_0$$

$$= 0 \quad , \quad \text{ow}$$

& so

$$f_{(m)}(x) = \frac{m}{t_0}\left(\frac{x}{t_0}\right)^{m-1}, \quad 0 \le x \le t_0$$
$$= 0, \quad ow$$

If $t_0 = 1$ then
$$f_{(m)}(x) = m\, x^{m-1}, \quad 0 \le x \le 1$$
$$= 0, \quad ow$$

Remark

$$r(x) = E(Y \mid X = x) = \sum_y y\, f(y \mid x)$$

Y, X discrete

$$f(y \mid x) = \frac{f(x, y)}{f(x)}$$

$f(x,y) = P(X = x, Y = y)$, $f(x) = P(X = x)$

$$\boxed{E(Y \mid X) = r(X)}$$

$$\boxed{E[E(Y \mid X)] = E(Y)}$$

$$E[E(z^Y \mid X)] = E(z^Y)$$

can be extended to the cts case & then

$$r(x) = E(Y \mid X = x) = \int_{-\infty}^{\infty} y\, \underbrace{f(y \mid x)}_{\text{cond pdf}}\, dy$$

stochastic process = collection of random elements

Gaussian process = collection of rv's $\ni$ dist'ns are normal

Markov property

$\{X_t : t \in T\}$

t — time; T — index set

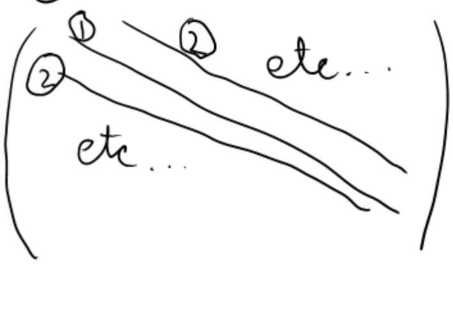
For each $t_0 \in T$,

$$g(\{X_t : t > t_0\}) \mid \{X_t : t \le t_0\}$$

$$\text{"="} \; g(\{X_t : t > t_0\}) \mid X_{t_0}, \quad \text{"}\forall g\text{"}$$

stationary process: $\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_k} \end{pmatrix} \overset{d}{=} \begin{pmatrix} X_{t_1+\Delta} \\ \vdots \\ X_{t_k+\Delta} \end{pmatrix}$, "$\forall$" $\underset{\sim}{t}$, Δ

Toeplitz matrix



elements on 1 same

" " 2 same

square

Poisson processes & order stats

<u>sample</u> $X_1, \ldots, X_n$

<u>order stats</u> $X_{(1)} < \cdots < X_{(n)}$

(assume <u>cts</u>)

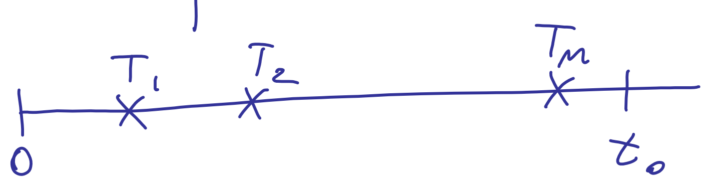
Poisson process

- throw points $\to$ Poisson process
- have process via some other way & condition $\to$ throw pts!

<u>So</u>

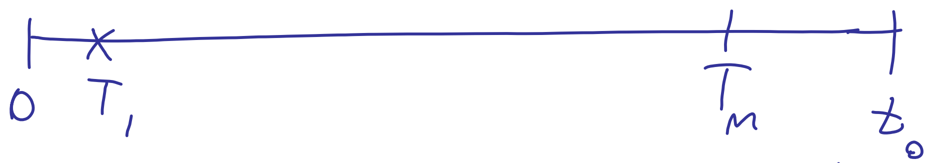
Let $\{N(t) : t \geq 0\}$ be a Poisson process of rate λ on $t \geq 0$.

Suppose you <u>know</u> $N(t_0) = n$. Call the times of the points $T_1 < T_2 < \cdots < T_n$

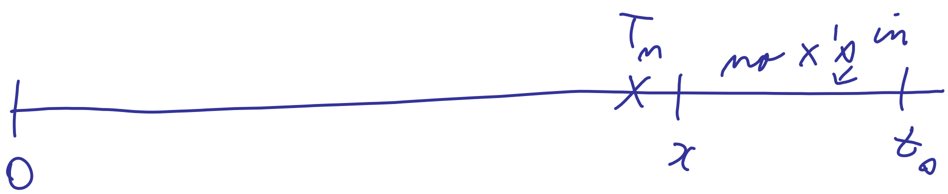


We want the pdf of T_n.

Back to our Poisson process.



$$F_{(n)}(x) = P(T_n \le x) \quad \left\{ = P\left(T_n \le x \,\middle|\, N(t_0) = n\right) \right\}$$



$\equiv$ } conditional calculations

$$
\begin{aligned}
F_{(n)}(x) &= \left(\frac{x}{t_0}\right)^n, && 0 \le x \le t_0 \\
&= 1, && x > t_0 \\
&= 0, && \text{o w}
\end{aligned}
$$

& so

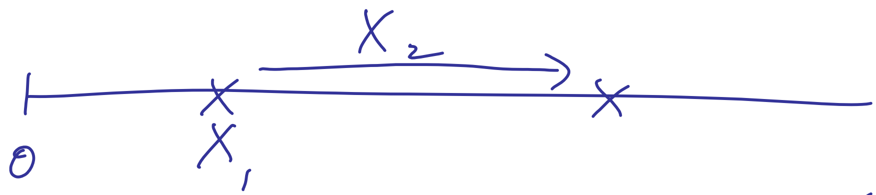
$$f_{(n)}(x) = \frac{n}{t_0}\left(\frac{x}{t_0}\right)^{n-1}, \quad 0 \le x \le t_0$$
$$= 0, \quad ow$$

If $t_0 = 1$ then
$$f_{(n)}(x) = n\, x^{n-1}, \quad 0 \leq x \leq 1$$
$$= 0, \quad ow$$

As we will see this is what we get when looking at the order statistics from a uniform.

First Interarrival times ind?

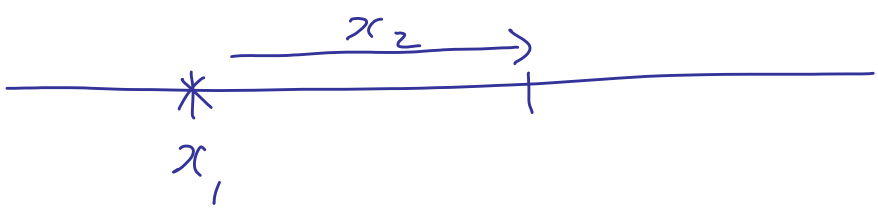
conditional pdf / pf / mgf / pgf / df = unconditional $\Rightarrow$ ind eg Poisson process, rate λ on $t \geq 0$



We know $X_1 \sim$ exponential(λ)

Look at

$$P(X_2 > x_2 \mid X_1 = x_1)$$



$$= P\big(\underbrace{N((x_1, x_1 + x_2]) = 0}\ \big|\ \underbrace{X_1 = x_1}\big)$$

ind

$$= P\big(N((x_1, x_1 + x_2] = 0)\big)$$

$$= e^{-\lambda x_2}$$

which is the tail probability f'n of an

exponential(λ).

$\therefore X_1$ & X_2 are independent & the unconditional tail probability f'n is as above. That is, X_1, X_2 are iid exponential(λ). Continue to get $X_1, X_2, \ldots$ iid exponential

Order Stats

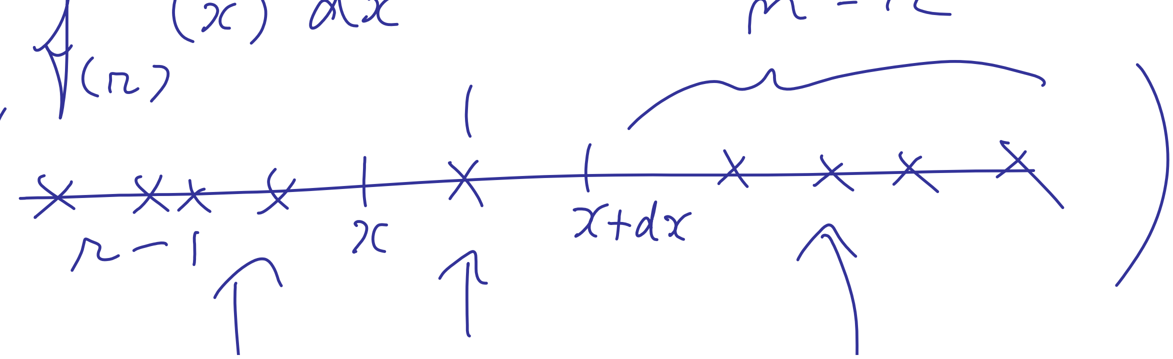
$X_1, X_2, \ldots, X_n$ iid pdf f / df F

$$X_{(1)} < \cdots < X_{(n)} \Leftarrow \text{order stats}$$

pdf of $X_{(r)}$

$$f_{(r)}(x)\,dx$$

$\approx P\Big($



$\Big)$

$$F(x) \quad \approx f(x)dx \quad \approx \bar{F}(x)$$

$$\approx \binom{n}{r-1,\, 1,\, n-r} F(x)^{r-1} f(x)\,dx \; \bar{F}(x)^{n}$$

"

$$f_{(r)}(x) = \binom{n}{r-1,\, 1,\, n-r} F(x)^{r-1} f(x)\, \bar{F}(x)^{n}$$

For a uniform $(0,1)$

$$f_{(n)}(x) = n\, x^{n-1}, \quad 0 < x < 1$$