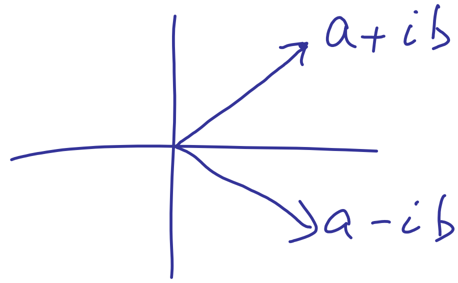
$$e^{i\theta} = \cos\theta + \overset{\sqrt{-1}}{\overset{\downarrow}{i}} \sin\theta$$



$$\overline{e^{i\theta}} = e^{-i\theta}$$

$$a - ib = \overline{a+ib}$$

$$(a+ib)\overline{(a+ib)} = a^2+b^2 = |a+ib|^2$$

$$\left|e^{i\theta_2} - e^{i\theta_1}\right| \leq |\theta_2 - \theta_1|$$

<u>cf</u> $c(t) = E\left(e^{itX}\right) = E(\cos tX) + i\,E(\sin tX)$

<u>Note</u> $c(0) = 1$

$c_{X+Y}(t) = c_X(t)\,c_Y(t)$ if X & Y ind

eg $X \sim$ Cauchy

$$f(x) = \frac{1}{\pi(1+x^2)}$$

$$E(\sin tX) = \int_{-\infty}^{\infty} \frac{\sin tx}{\pi(1+x^2)}\,dx = 0$$

$$E(\cos tX) = \int_{-\infty}^{\infty} \frac{\cos tx}{\pi(1+x^2)}\,dx \underset{\substack{\uparrow \\ \text{tricky}}}{=} e^{-|t|}$$

$$c(t) = e^{-|t|}$$

Let $Y = -X$. Then $c_Y(t) = E\left(e^{itY}\right)$

$$= E\left(e^{it(-1)X}\right) = c_X(-t)$$

$$= e^{-|t|}$$

Look at $Y = a + bX$

$$\Rightarrow \quad c_Y(t) = e^{iat} c_X(bt)$$

<u>Theorem</u> $c(t)$, $-\infty < t < \infty$ determines the dist'n.

<u>Theorem</u> $m(t) = E(e^{tX})$ determines the dist'n if it is finite in a neighborhood of $t = 0$.

<u>Remark</u> In the cts case the <u>Inversion Theorem</u> gives us a formula for $f(x)$ in terms of $c(t)$. Not so nice for mgf's.

$$\left.\begin{array}{l} X \sim \text{Cauchy} \\ Y = X \end{array}\right\} \quad W = X + Y$$

$$c_W(t) = c_{2X}(t) = c_X(2t) = e^{-2|t|}$$

$$c_X(t)\, c_Y(t) = e^{-2|t|} \text{ in this case}$$

so that <u>in this case</u> $c_{X+Y}(t) = c_X(t)\, c_Y(t)$

but X & Y are not independent.

rvec $\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix}$ $\quad \overbrace{\underset{\sim}{t}'\underset{\sim}{Y}} = t_1 Y_1 + \cdots + t_m Y_m$

$$c_{\underset{\sim}{Y}}(\underset{\sim}{t}) = E\left[e^{i(\underset{\sim}{t}\cdot\underset{\sim}{Y})}\right]$$

$$= E\left[e^{i t_1 Y_1} \cdots e^{i t_m Y_m}\right]$$

<u>Theorem</u> If $c_{\underset{\sim}{Y}}(\underset{\sim}{t}) = c_{Y_1}(t_1) \cdots c_{Y_m}(t_m)$, $\forall \underset{\sim}{t}$

then $Y_1, \ldots, Y_m$ are independent

<u>Note</u> Also true for mgf's $m_{\underset{\sim}{Y}}(\underset{\sim}{t}) = E(e^{\underset{\sim}{t}\cdot\underset{\sim}{Y}})$

" " " df's

$$F_{\underset{\sim}{Y}}(y) = P(\underset{\sim}{Y} \le y) = \underbrace{P(Y_1 \le y_1)}_{F_{Y_1}(y_1)} \cdots \underbrace{P(Y_m \le y_m)}_{F_{Y_m}(y_m)}, \forall y$$

" " ~~for pgf's~~

If $\underset{\sim}{Y}$ is cts then $\exists$ a pdf $f(y)$ st

$$E[\underset{\text{real}}{g}(\underset{\sim}{Y})] = \int g(y) f(y)\, dy$$

<u>Note</u> Knowing $E[g(\underset{\sim}{Y})]$, $\forall g$, determines the dist'n.

So if we have 2 pdf's $f_1(y)$ & $f_2(y)$ s.t

$$\int g(y) f_1(y)\, dy = \int g(y) f_2(y)\, dy \quad , \forall g$$

$$\Rightarrow \quad f_1 = f_2$$

If $f(y) = f_1(y_1) \cdots f_m(y_m)$

$$\Rightarrow E\left[g_1(Y_1) \cdots g_m(Y_m)\right] = \int g_1(y_1) \cdots g_m(y_m) f_1(y_1) \cdots f_m(y_m)\, dy$$

$$= \int g_1(y_1) f(y_1)\, dy_1 \cdots \int g_m(y_m) f_m(y_m)\, dy_m$$

$$= E(g_1(Y_1)) \cdots E(g_m(Y_m)) \quad , \text{"}\forall\text{"}\ g_1, \dots, g_m$$

$\Rightarrow Y_1, \dots, Y_m$ are ind

(also true in the discrete case)

$$F(y) = P(Y_1 \le y_1, \dots, Y_m \le y_m)$$

In the cts case this will be an integral &

$$\frac{\partial^m}{\partial y_1 \cdots \partial y_m} F(y) = f(y) \qquad (*)$$

& if $F(y) = F_1(y_1) \cdots F_m(y_m)$ & $(*)$ holds

then $f(y) = f_1(y_1) \cdots f_m(y_m) \Rightarrow$ independence

$$\left.\begin{array}{l} E(e^{tY} \mid X) = E(e^{tY}) \\ E(e^{itY} \mid X) = E(e^{itY}) \\ F(y \mid X) = F(y) \\ \quad \text{etc} \\ \quad \vdots \end{array}\right\} \Rightarrow \text{ind of } X \,\&\, Y$$

Can use this to show that

$X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$

$$\Rightarrow \bar{X} \;\&\; S^2 = \sum (X_i - \bar{X})^2$$

are ind (see web for the calculation)

Change of variables

$\underset{m\times 1}{\underset{\sim}{X}}$ & $\underset{m\times 1}{\underset{\sim}{Y}} = \underset{1-1}{h}(\underset{\sim}{X})$ then

$$f_{\underset{\sim}{Y}}(y) = f_{\underset{\sim}{X}}(h^{-1}(y)) \left| \det\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}'}\right) \right| ,$$

where $\dfrac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}'} = \underset{m\times m}{\left(\dfrac{\partial \underset{\sim}{x}}{\partial y_1} \;\; \dfrac{\partial \underset{\sim}{x}}{\partial y_2} \;\; \cdots \;\; \dfrac{\partial \underset{\sim}{x}}{\partial y_m}\right)}$ ← Jacobian matrix

$$E[g(\tilde{Y})] = \int g(\tilde{y})\, f_{\tilde{Y}}(\tilde{y})\, d\tilde{y}$$

$\forall g$

$$\|$$

$$E[g\circ h(\tilde{X})] = \int g(h(\tilde{x}))\, f_{\tilde{X}}(\tilde{x})\, d\tilde{x}$$

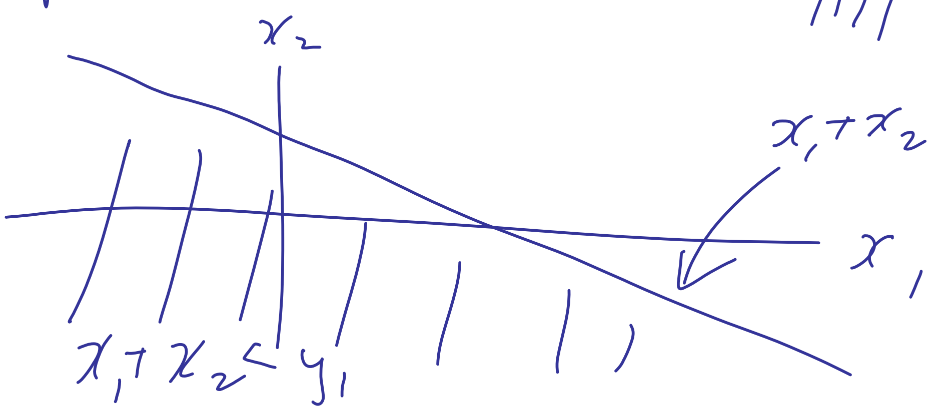
$$\overset{\tilde{u}=h(\tilde{x})}{=} \int g(\tilde{u})\, \underbrace{f_{\tilde{X}}(h^{-1}(\tilde{u})) \left|\det\left(\frac{\partial \tilde{x}}{\partial \tilde{u}'}\right)\right|}\, d\tilde{u}$$

& so we our formula.

eg X_1 & X_2 ind & $Y_1 = X_1 + X_2$

$X_1 \downarrow f_1(x_1)$ $\quad$ $X_2 \downarrow f_2(x_2)$

$$P(Y_1 \le y_1) = \int \overbrace{f(x_1, x_2)}^{f_1(x_1) f_2(x_2)}\, dx_1 dx_2$$



get $F(y_1)$ & then $f(y_1) = F'(y_1)$.

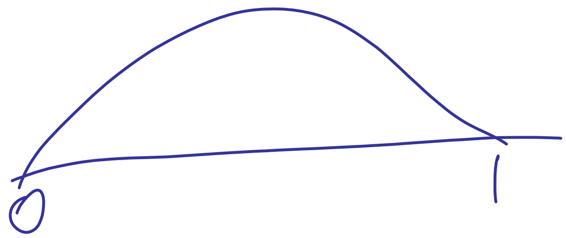
<u>Or</u>

$$\left.\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 \end{aligned}\right\} \quad \tilde{X} \to \tilde{Y}$$

- apply change of variable
- $f(y_1, y_2) \Rightarrow f(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2$

Problem: $X \sim \text{gamma}(r_1, \lambda)$, ind $\to$ $Y \sim$ " (r_2, λ) $\Big\}$ $W = \dfrac{X}{X+Y}$

Then W is a beta rv.



$Z \sim N(0,1)$; $E(e^{tZ}) = e^{t^2/2}$

$\Rightarrow$ pdf is $f(z) \propto e^{-z^2/2}$

$Y \sim N(\mu, \sigma^2)$; $E(e^{tY}) = e^{\mu t} e^{\sigma^2 t^2/2}$

$\xRightarrow{\sigma>0} f(y) \propto e^{-\left(\frac{y-\mu}{\sigma}\right)^2/2}$

$Y \overset{d}{=} \mu + \sigma \underset{N(0,1)}{Z}$

Now the dist'n of $\underset{\sim}{Y}$ is determined by the dist'n of all $\underset{\sim}{c}'\underset{\sim}{Y}$.

Lemma If $\underset{\sim}{X}$ has mean $\underset{\sim}{\mu} = E(\underset{\sim}{X})$ & $\Sigma = \text{Var}(\underset{\sim}{X}) = \{\text{cov}(X_i, X_j)\}$ then

(1) $E(A\underset{\sim}{X} + \underset{\sim}{b}) = A E(\underset{\sim}{X}) + \underset{\sim}{b}$

(2) $\text{Var}(A\underset{\sim}{X} + \underset{\sim}{b}) = A \,\text{Var}(\underset{\sim}{X})\, A'$

prove it

Def'n $\underset{\sim}{Y}$ is $N(\underset{\sim}{\mu}, \Sigma)$ if

$\underset{\sim}{c}'\underset{\sim}{Y}$ is $N(\underset{\sim}{c}'\underset{\sim}{\mu}, \underset{\sim}{c}'\Sigma\underset{\sim}{c}), \forall \underset{\sim}{c}$

Theorem $\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \Sigma)$

$\Rightarrow A\underset{\sim}{Y} + \underset{\sim}{b} \sim N(A\underset{\sim}{\mu} + \underset{\sim}{b}, A\Sigma A')$

Proof: Easy ← do it

mgf of a $N(\underset{\sim}{\mu}, \Sigma)$

$\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \Sigma)$

$m(\underset{\sim}{t}) = E\left(e^{\underset{\sim}{t}'\underset{\sim}{Y}}\right)$

Now $\underset{\sim}{t}'\underset{\sim}{Y} \sim N(\underset{\sim}{t}'\underset{\sim}{\mu}, \underset{\sim}{t}'\Sigma\underset{\sim}{t})$

$\Rightarrow E\left(e^{\underset{\sim}{t}'\underset{\sim}{Y}}\right) = e^{\underset{\sim}{t}'\underset{\sim}{\mu}} \, e^{\underset{\sim}{t}'\Sigma\underset{\sim}{t}/2}$

Now $\Sigma = TT'$ $\left(\Sigma^{\frac{1}{2}}\ \Sigma^{\frac{1}{2}}\right)$ ← diag elements ≥ 0

$\Sigma = QDQ' = \underbrace{QD^{\frac{1}{2}}Q'}_{\Sigma^{1/2}}QD^{\frac{1}{2}}Q'$

Take $Z_1, \ldots, Z_n$ to be $N(0,1)$ & iid then if $\underset{\sim}{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}$ we have $\underset{\sim}{\mu} + T\underset{\sim}{Z} \sim N(\underset{\sim}{\mu}, \Sigma)$

The pdf of $\underset{\sim}{Z}$ is $\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-|\underset{\sim}{z}|^2/2}$ & if Σ has an inverse then so does T & then the pdf of

$$\underset{\sim}{W} = \underset{\sim}{\mu} + T\underset{\sim}{Z}$$

& so $\underset{\sim}{z} = T^{-1}(\underset{\sim}{w} - \underset{\sim}{\mu}) \Rightarrow \frac{\partial \underset{\sim}{z}}{\partial \underset{\sim}{w}'} = T^{-1}$

$$\Rightarrow \det\left(\frac{\partial \underset{\sim}{z}}{\partial \underset{\sim}{w}'}\right) = \frac{1}{\det(T)} = \frac{1}{\sqrt{\det \Sigma}}$$

Now use the change of variables formula to get

$$f(\underset{\sim}{y}) \propto e^{-(\underset{\sim}{y}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\mu})/2}$$

Now let $\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \Sigma)$. Then

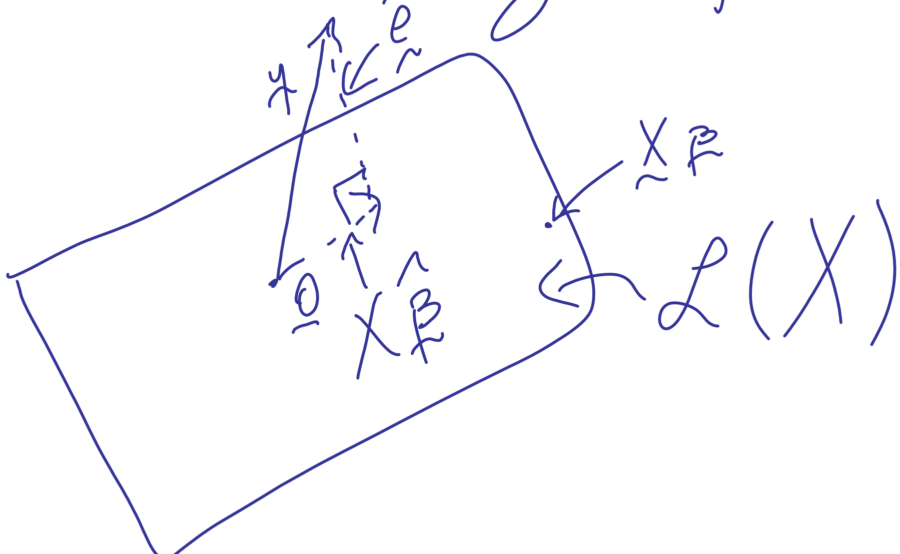
$$m_{\underset{\sim}{Y}}(\underset{\sim}{t}) = e^{\underset{\sim}{\mu}'\underset{\sim}{t}}\, e^{\underset{\sim}{t}'\Sigma\underset{\sim}{t}/2}$$

so that Σ diagonal $\Rightarrow$ the components are independent eg $\underset{\sim}{y} = \underbrace{X}_{n \times r \text{ \& known}} \underbrace{\underset{\sim}{\beta}}_{r \times 1} + \underset{\sim}{e} \qquad \underset{\sim}{e} \sim N(\underset{\sim}{0}, \sigma^2 I)$

$y_1, \ldots, y_n$ iid $N(\mu, \sigma^2)$. Then

$$\underset{\sim}{y} = \underbrace{\underset{\sim}{1}}_{X} \underbrace{\mu}_{\underset{\sim}{\beta}\ (r=1)} + \underset{\sim}{e}$$

Choose $\hat{\underset{\sim}{\beta}}$ to minimize $|\underset{\sim}{y} - X\underset{\sim}{\beta}|^2$



$$X\hat{\underset{\sim}{\beta}} = \underbrace{X(X'X)^{-1}X'}_{\text{projection matrix} \leftarrow \text{call it } P} \underset{\sim}{y}$$

Notice $P = P'$, $P = P^2$. It is idempotent. Also, since $\underset{\sim}{c}'P\underset{\sim}{c} \geq 0$ (clearly?) we have $P = QDQ'$ where $QQ' = Q'Q = I$ & D is diagonal with ≥ 0 diagonal elements

Now $P^2 = QDQ'QDQ'$
$= QD^2Q' = P = QDQ'$
so that $QD^2Q' = QDQ' \Rightarrow D^2 = D$ & hence the diagonal terms of D are 0's & 1's. The # of 1's will be the rank of P which in this case is also the rank of X & of $X'X$.

Set $\hat{\underset{\sim}{y}} = X\hat{\underset{\sim}{\beta}} = P\underset{\sim}{y}$. Then $\hat{\underset{\sim}{e}} = \underset{\sim}{y} - \hat{\underset{\sim}{y}} = (I-P)\underset{\sim}{y}$.
$I-P$ is the projection matrix for $\mathcal{L}(X)^{\perp}$ – the subspace $\perp$ to $\mathcal{L}(X)$. If X has rank r then $I-P$ will have rank $n-r$. Now

$$\begin{pmatrix} \hat{\underset{\sim}{y}} \\ \hat{\underset{\sim}{e}} \end{pmatrix} = \begin{pmatrix} P \\ I-P \end{pmatrix} \underset{\sim}{y}$$

so that this is a vector normal. We can calculate its variance to be

$$\mathrm{Var}\begin{pmatrix} \hat{\underset{\sim}{y}} \\ \hat{\underset{\sim}{e}} \end{pmatrix} = \sigma^2 \begin{pmatrix} P^2 & 0 \\ 0 & (I-P^2) \end{pmatrix} = \sigma^2 \begin{pmatrix} P & 0 \\ 0 & I-P \end{pmatrix}$$

& hence $\hat{\underset{\sim}{y}}$ & $\hat{\underset{\sim}{e}}$ are independent. In the special case of a sample from a $N(\mu, \sigma^2)$ we have

$$\hat{\underset{\sim}{y}} = \bar{y}\underset{\sim}{1} \quad \& \quad \hat{\underset{\sim}{e}} = \underset{\sim}{y} - \bar{y}\underset{\sim}{1}$$

being independent so that $\bar{y}$ and $|\underset{\sim}{y} - \bar{y}\underset{\sim}{1}|^2 = \sum_{k=1}^{n}(y_k - \bar{y})^2$ are independent.

Notice $|\hat{\underset{\sim}{e}}|^2 = |(I-P)\underset{\sim}{y}|^2 = |QDQ'\underset{\sim}{y}|^2$, where

D has $n-r$ diagonal 1's & r diagonal 0's. Since

$$(I-P)\underset{\sim}{y} = (I-P)X\underset{\sim}{\beta} + (I-P)\underset{\sim}{e} = \underset{\sim}{0} + (I-P)\underset{\sim}{e} = (I-P)\underset{\sim}{e}$$

we have

$$|\hat{\underset{\sim}{e}}|^2 = |QDQ'\underset{\sim}{e}|^2 = \underset{\sim}{e}'QDQQDQ'\underset{\sim}{e}$$

$$= \underset{\sim}{e}'D\underset{\sim}{e} \qquad (D^2 = D)$$

$$\stackrel{d}{=} \sigma^2(Z_1^2 + \cdots + Z_{n-r}^2),$$

where $Z_1, \cdots, Z_{n-r}$ are iid $N(0,1)$. Hence

$$\frac{|\hat{\underset{\sim}{e}}|^2}{\sigma^2} \stackrel{d}{=} Z_1^2 + \cdots + Z_{n-r}^2$$

& this is a $\chi^2(n-r)$ rv. In the special problem of a sample of size n from a $N(\mu, \sigma^2)$ we have $\bar{y} \sim N(\mu, \sigma^2/n)$ & this is independent of $\sum_{k=1}^{n}(y_k-\bar{y})^2/\sigma^2$ which is $\chi^2(n-1)$. If we set $s^2 = \sum_{k=1}^{n}(y_k-\bar{y})^2/(n-1)$ we then have

$$\frac{\bar{y}-\mu}{s/\sqrt{n}} \sim t(n-1)$$

<u>Remarks</u> ① An alternative approach to showing the independence of $\bar{y}$ & s^2 when sampling from a normal uses conditional mgf's & a change of variables. It is given below.

② A derivation of the pdf of a chi-squared rv using the df & its derivative may be found at the end of this document.

eg Let $X_1, \cdots, X_n$ be iid $N(\mu, \sigma^2)$.
Then

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n} \quad \& \quad S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$$

are independent.

Sol'n We will calculate $E(e^{tS^2} | \bar{x})$ & show it does not depend on $\bar{x}$.

We know $\bar{X} \sim N(\mu, \sigma^2/n)$. Now look at the jt pdf of $\bar{X}, X_2, \ldots, X_n$.

We have

$$\bar{x} = \frac{x_1 + \cdots + x_n}{n}$$

$$x_2 = x_2$$

$$\vdots$$

$$x_n = x_n$$

$$\Rightarrow \quad x_1 = n\bar{x} - (x_2 + \cdots + x_n)$$

$$x_2 = x_2$$

$$\vdots$$

$$x_n = x_n$$

$$\Rightarrow \quad \frac{\partial \underset{\sim}{x}}{\partial(\bar{x}, x_2, \ldots, x_n)} = \begin{pmatrix} n & -1 & - & - & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ & & & \ddots & \\ 0 & \cdots & & & 1 \end{pmatrix}$$

$$\Rightarrow \det\left(\quad \right) = n$$

The pdf of $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ is

$$f(\underset{\sim}{x}) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

<u>Now</u> $\displaystyle\sum_{i=1}^{n}(x_i-\mu)^2 = \underbrace{\sum_{i=1}^{n}(x_i-\bar{x})^2}_{s^2} + n(\bar{x}-\mu)^2$

$$\Rightarrow (x_1-\mu)^2 = s^2 + n(\bar{x}-\mu)^2 - \sum_{i=2}^{n}(x_i-\mu)^2$$

$\therefore$ $f(\bar{x}, x_2, \ldots, x_n)$

$$= f\left(\underset{\sim}{x} \text{ in terms of } \begin{pmatrix} \bar{x} \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) n$$

$$= n f(n\bar{x} - x_2 - \cdots - x_n, x_2, \ldots, x_n)$$

$$= n\, f(n\bar{x} - x_2 - \cdots - x_n)\, f(x_2) \cdots f(x_n)$$

$$= n\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\left(s^2 + n(\bar{x}-\mu)^2\right)\right) \quad (*)$$

$$\Rightarrow f(x_2, \cdots, x_n \mid \bar{x})$$

$$= \frac{f(\bar{x}, x_2, \cdots, x_n)}{f(\bar{x})}$$

$$= \frac{(*)}{\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma/\sqrt{n}} \exp\left(-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right)}$$

$$= \frac{n\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n}{\frac{1}{\sqrt{2\pi}\,\sigma/\sqrt{n}}} \exp\left(-\frac{s^2}{2\sigma^2}\right)$$

$$= \underbrace{\frac{\sqrt{n}}{(\sqrt{2\pi}\,\sigma)^{n-1}}}_{c} \exp\left(-\frac{s^2}{2\sigma^2}\right)$$

$$\Rightarrow \int \cdots \int e^{-\frac{s^2}{2\sigma^2}} dx_2 \cdots dx_n = \frac{(\sqrt{2\pi}\,\sigma)^{n-1}}{\sqrt{n}}$$

Now

$$E\left(e^{t S^2/\sigma^2} \,\middle|\, \bar{x}\right)$$

$$= \int \cdots \int e^{t s^2/\sigma^2} c\, e^{-\frac{s^2}{2\sigma^2}} dx_2 \cdots dx_n$$

$$= c \int \cdots \int \exp\left(-\frac{s^2}{2\sigma^2}(1-2t)\right) dx_2 \cdots dx_n$$

$$= c \int \int \exp\left(-\frac{s^2}{2\sigma_t^2}\right) dx_2 \cdots dx_n,$$

where $\sigma_t^2 = \frac{\sigma^2}{1-2t}$ & so we get

$$c\, \frac{(\sqrt{2\pi}\,\sigma_t)^{n-1}}{\sqrt{n}} = \frac{\sqrt{n}}{(\sqrt{2\pi}\,\sigma)^{n-1}} \frac{(\sqrt{2\pi}\,\sigma_t)^{n-1}}{\sqrt{n}}$$

$$= \left(\frac{\sigma_t}{\sigma}\right)^{n-1} = \left(\frac{1}{\sqrt{1-2t}}\right)^{n-1}$$

so that $\frac{S^2}{\sigma^2}$ & $\bar{X}$ are ind & hence so are S^2 & $\bar{X}$.

<u>Remark</u> The above also yields

$$E\left(e^{t\, S^2/\sigma^2}\right) = \left(\frac{1}{\sqrt{1-2t}}\right)^{n-1}, \ t<\frac{1}{2}$$

which is the mgf of a $\chi^2(n-1)$.

r vec $\underset{\sim}{Y}$

mgf of $\underset{\sim\, m\times 1}{Y}$ is

$$m(\underset{\sim}{t}) = E\left(e^{\underset{\sim}{t}'\underset{\sim}{Y}}\right)$$

Fact – $m(\underset{\sim}{t})$ determines the dist'n if it exists in an open neighborhood of $\underset{\sim}{0}$

– $m(\underset{\sim}{t}) = f_m(t_1) \cdots f_m(t_m)$
$\Leftrightarrow$ components of $\underset{\sim}{Y}$ are independent

Let $\underset{m\times m}{\Sigma}$ be a variance matrix.

Let $Z_1, \ldots, Z_m$ be iid $N(0,1)$ &

$$\underset{\sim}{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_m \end{pmatrix}$$

$$\Sigma = \underset{\perp}{Q} D Q' = \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} = TT'$$

$\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \Sigma)$ if

$$\underset{\sim}{Y} \overset{d}{=} \mu + \Sigma^{\frac{1}{2}} \underset{\sim}{Z}$$

If Σ^{-1} exists then we can easily use a change of variables to get

$$f(y) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}(y-\underset{\sim}{\mu})' \Sigma^{-1}(y-\underset{\sim}{\mu})}, \quad \forall y$$

The mgf of $\underset{\sim}{Y}$ is

$$E\left(e^{\underset{\sim}{t}'\underset{\sim}{Y}}\right) = E\left(e^{t'\underset{\sim}{\mu} + t'\Sigma^{\frac{1}{2}}\underset{\sim}{Z}}\right)$$

$$= e^{\underset{\sim}{t}'\underset{\sim}{\mu}} e^{\underset{\sim}{t}'\Sigma \underset{\sim}{t}/2}$$

<u>Prop</u> $\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \Sigma)$

$$\Rightarrow A\underset{\sim}{Y} + \underset{\sim}{b} \sim N(A\underset{\sim}{\mu} + \underset{\sim}{b}, A\Sigma A')$$

Pf Easy — use mgf's

So the mgf of a $N(\underset{\sim}{\mu}, \Sigma)$ is

$$m(\underset{\sim}{t}) = e^{\underset{\sim}{t}'\underset{\sim}{\mu}} \, e^{\underset{\sim}{t}'\Sigma\underset{\sim}{t}/2}$$

which "factors out" if $\text{cov}(Y_i, Y_j) = 0$ for $i \neq j$. So for normal rvec's independence & uncorrelated are the same.

<u>chi-squared</u> χ^2

$$\underset{n \times 1}{\underset{\sim}{Z}} \sim N(\underset{\sim}{0}, I); \; f(\underset{\sim}{z}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-|\underset{\sim}{z}|^2/2}$$

If $Y \stackrel{d}{=} |\underset{\sim}{Z}|^2$ then we say Y is chi-squared with n degrees of freedom. We want to derive its pdf.

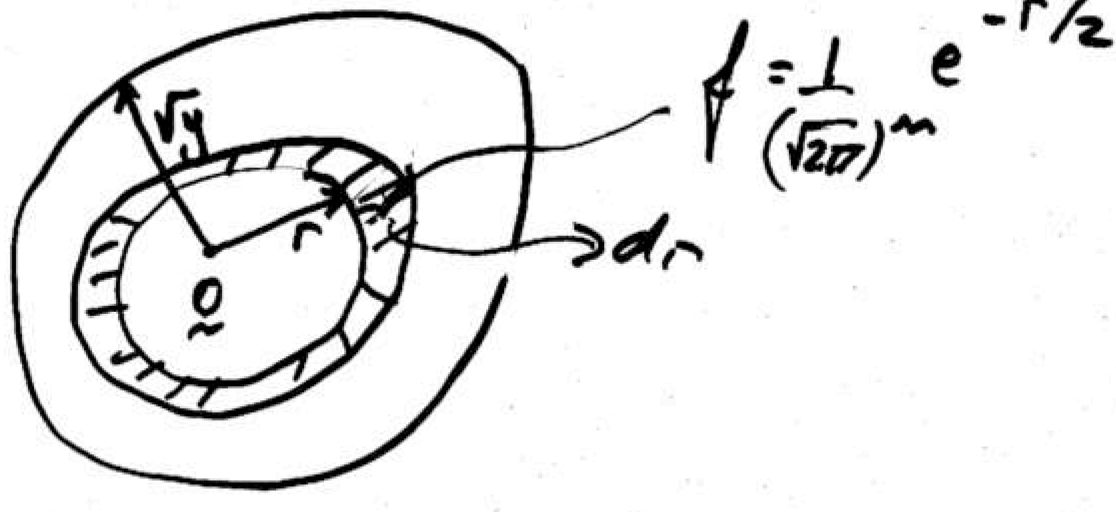
Sol'n Let $Z_1, \cdots, Z_n$ be iid $N(0,1)$; $\underset{\sim}{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}$; $Y = |\underset{\sim}{Z}|^2$

We know that the volume of a ball (sphere) of radius $r = c r^n$

$\Rightarrow$ surface area $= n c r^{n-1}$.

Let $y > 0$ then

$$F(y) = P(Y \le y)$$
$$= P(|\underset{\sim}{Z}| \le \sqrt{y})$$



$$f(\underset{\sim}{z}) = \left(\frac{1}{\sqrt{2\pi}}\right)^m e^{-|\underset{\sim}{z}|^2/2}$$

$$F(y) = \int_0^{\sqrt{y}} \frac{1}{(\sqrt{2\pi})^m} e^{-r^2/2} m c r^{m-1} dr$$

$$\Rightarrow f(y) = \frac{1}{(\sqrt{2\pi})^m} e^{-y/2} m c (\sqrt{y})^{m-1} \frac{1}{2\sqrt{y}}$$

$$= \frac{c\,m}{2(\sqrt{2\pi})^m} e^{-y/2} \; y^{\frac{m}{2}-1}$$

Now $\int_0^\infty f(y)\,dy = 1$

$$\Rightarrow \int_0^\infty c \frac{m}{2} \left(\frac{1}{\sqrt{2\pi}}\right)^m y^{\frac{m}{2}-1} e^{-y/2} dy = 1$$

$$\overset{u=y/2}{\Rightarrow} \int_0^\infty c\,m \left(\frac{1}{\sqrt{2\pi}}\right)^m (2u)^{\frac{m}{2}-1} e^{-u} du = 1$$

$$\Rightarrow \frac{c\, n\, 2^{\frac{n}{2}-1}}{\left(\sqrt{2\pi}\right)^n} \underbrace{\int_0^\infty u^{\frac{n}{2}-1} e^{-u}\, du}_{\Gamma\left(\frac{n}{2}\right)} = 1$$

$$\Rightarrow c = \frac{(2\pi)^{n/2}}{2^{\frac{n}{2}-1}\, n\, \Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{\pi^{n/2}}{\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

= volume of a ball/sphere of radius 1 in n-dim

Also,

$$Y \sim \text{gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

Another Take on the Normal

$$X \sim N(\mu, \sigma^2) \Leftrightarrow m(t) = e^{\mu t + \sigma^2 t^2/2}$$
$$\Leftrightarrow c(t) = e^{i\mu t - \sigma^2 t^2/2}$$

$$\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \Sigma)$$
$$\Leftrightarrow \underset{\sim}{c}' \underset{\sim}{Y} \sim N(\underset{\sim}{c}'\underset{\sim}{\mu}, \underset{\sim}{c}' \Sigma \underset{\sim}{c})$$
$$\Leftrightarrow m(\underset{\sim}{t}) = E\left(e^{\underset{\sim}{t}'\underset{\sim}{Y}}\right) = e^{\underset{\sim}{\mu}'\underset{\sim}{t} + \underset{\sim}{t}'\Sigma\underset{\sim}{t}/2}$$
$$\Leftrightarrow c(\underset{\sim}{t}) = e^{i\underset{\sim}{\mu}'\underset{\sim}{t} - \underset{\sim}{t}'\Sigma\underset{\sim}{t}/2}$$

<u>Theorem</u>

1. $\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \Sigma) \Rightarrow A\underset{\sim}{Y} + \underset{\sim}{b} \sim N(A\underset{\sim}{\mu} + b, A\Sigma A')$

2. Let $\underset{\sim}{Y} = \begin{pmatrix} \underset{\sim}{Y}_1 \\ \underset{\sim}{Y}_2 \end{pmatrix} \sim N(\underset{\sim}{\mu}, \Sigma)$

$\underset{\sim}{Y}_1$ & $\underset{\sim}{Y}_2$ ind $\Leftrightarrow \Sigma_{12} = 0$,

where $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

Theorem Suppose $\underset{\sim}{Y} \sim N(\underset{\sim}{\mu}, \Sigma)$ where Σ^{-1} exists. Then $\underset{\sim}{Y}$ has pdf

$$(*) \quad f(y) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left[-\frac{1}{2}(\underset{\sim}{y}-\underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{y}-\underset{\sim}{\mu})\right]$$

Proof Σ^{-1} exists $\Rightarrow \Sigma = TT'$ with T triangular + >0 diagonal terms (so that $\det(T) > 0$ $(= \sqrt{\det(\Sigma)})$). Now if

$$\underset{\sim}{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \leftarrow \text{iid } N(0,1) \text{ then}$$

$$\underset{\sim}{\mu} + T\underset{\sim}{Z} \sim N(\underset{\sim}{\mu}, \Sigma)$$

Now use the change of variables formula to show $\mu + T\underset{\sim}{Z}$ has pdf $(*)$

qed