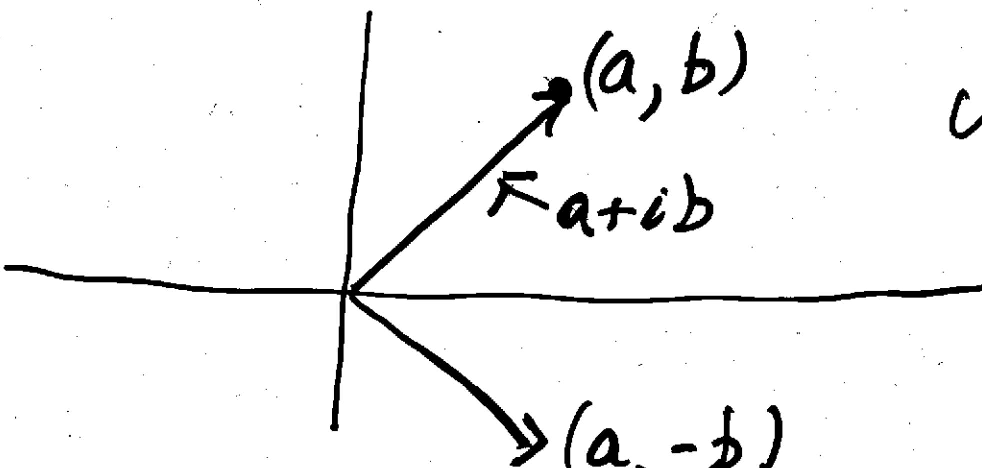
Some background

$a, b \in \mathbb{R}$ & i an object st $i^2 = -1$ then $a + ib$ is called a complex #.

a ↑ real part

b ↑ imaginary part



complex plane

complex conjugate $\overline{a+ib} = a - ib$

If $z = a + ib$ then $|z|^2 = a^2 + b^2$

Notice $z\bar{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$

$e^{it} = \cos t + i \sin t$ (def'n)

Note $e^z = e^a e^{ib}$

$$\overline{e^{it}} = \cos t - i \sin t$$

$$= \cos(-t) + i \sin(-t) = e^{-it}$$

$$e^{it_1} e^{it_2} = e^{i(t_1+t_2)}$$ - can check

$$|e^{it}|^2 = \cos^2 t + \sin^2 t = 1$$

$$\Rightarrow |e^{it}| = 1$$

$$|e^{it_2} - e^{it_1}| \le |t_2 - t_1|$$

$$|e^{it_2} - e^{it_1}| \le 2$$

<u>Note</u> $|z_1 + z_2| \le |z_1| + |z_2|$

$$|e^{it_2} - e^{it_1}| = \left| \int_{t_1}^{t_2} \frac{d(e^{it})}{dt} \, dt \right|$$

$$\le \left| \int_{t_1}^{t_2} \left| \frac{de^{it}}{dt} \right| dt \right|$$

$$= \left| \int_{t_1}^{t_2} \underbrace{|ie^{it}|}_{1} dt \right| = |t_2 - t_1|$$

Note, $|z_1 z_2| = |z_1||z_2|$

$\Rightarrow$ $|\cos t_2 - \cos t_1| \le |t_2 - t_1|$

$|\sin t_2 - \sin t_1| \le |t_2 - t_1|$

X rv

$$c(t) = E(e^{itX}) \quad \text{———} \quad \underline{\text{characteristic}}\ \underline{\underline{\text{c f}}}\ \text{f'n}$$

$$= E[\cos(tX)] + i\, E[\sin(tX)]$$

<u>Proposition</u>

(i) $c(0) = 1$, $|c(t)| \le 1$

(ii) $\overline{c(t)} = c(-t)$

(iii) $c_{aX+b}(t) = e^{itb} c_X(at)$

Inversion Theorem

(i) If X has pdf $f(x)$ then
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} c(t)\,dt$$

$$\left[\ c(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx\ \right]$$

(ii) $$F(y) - F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{-ity} - e^{-itx}}{-it}\right) c(t)\,dt$$

at ct pts of F

Note:- $c(t)$ determines the dist'n

$\exists\ \underset{\sim}{X}$ $\quad c(\underset{\sim}{t}) = E\left(e^{i\underset{\sim}{t}'\underset{\sim}{X}}\right)$

$\exists\ c(\underset{\sim}{t}) \Longleftrightarrow F(\underset{\sim}{x}) = P(\underset{\sim}{X} \le \underset{\sim}{x})$

Fact $F(\underset{\sim}{x}) = F(x_1)\cdots F(x_n) \Leftrightarrow$ independence

$c(\underset{\sim}{t}) = c(t_1)\cdots c(t_n) \Rightarrow F(\underset{\sim}{x}) = F(x_1)\cdots F(x_n) \Rightarrow$ ind

<u>Prop</u> X & Y ind $\Rightarrow c_{X+Y}(t) = c_X(t)\, c_Y(t)$

<u>Pf</u> $c_{X+Y}(t) = E\left[e^{it(X+Y)}\right]$

$$= E\left(e^{itX} e^{itY}\right)$$

$$= E\left(e^{itX}\right) E\left(e^{itY}\right)$$

<u>Note</u> $c_{X+Y}(t) = c_X(t)\, c_Y(t) \not\Rightarrow$ ind

<u>eg</u> X is Cauchy if

$$f(x) = \frac{1}{\pi(1+x^2)}$$

$$c(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)}\, dx = e^{-|t|}$$

Let $Y = X$. Then Y & X are not ind.

$$c_{X+Y}(t) = c_{2X}(t) = c_X(2t) = e^{-2|t|}$$

$$c_X(t)\, c_Y(t) = e^{-|t|} e^{-|t|} = e^{-2|t|}$$

& so $c_{X+Y}(t) = c_Y(t)\, c_Y(t)$

but Y & X are dependent.

<u>Remarks</u>

- $m(t) = E(e^{tX})$ — mgf
- $m(\underset{\sim}{t}) = E(e^{\underset{\sim}{t}'\underset{\sim}{X}})$ — mgf of $\underset{\sim}{X}$

<u>Fact</u>: If $m(\underset{\sim}{t})$ exists in a neighborhood of $\underset{\sim}{0}$ then $c(\underset{\sim}{t}) = m(i\underset{\sim}{t})$

eg $X \sim N(0,1) \iff m(t) = e^{t^2/2}, \ \forall t$

$$\Rightarrow c(t) = e^{(it)^2/2} = e^{-t^2/2}$$

2 A version of Taylor's Theorem

If $g^{(m)}(0)$ exists then

$$g(x) = \left[\sum_{j=0}^{m} \frac{g^{(j)}(0)}{j!} x^j\right] + o(x^m),$$

as $x \to 0$. Here $o(x^m)$ is the remainder & has the property that

$$\lim_{x \to 0} \frac{o(x^m)}{x^m} = 0$$

Also, $g^{(0)} = g$.

eg $e^x = 1 + x + o(x)$

$$e^{x + o(x)} = 1 + x + o(x)$$

not the same expression!

Formally,

$$e^{itX} = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} X^j$$

"& so"
$$E(e^{itX}) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} E(X^j)$$

"& so" $c^{(j)}(0) = i^j E(X^j)$

<u>Note</u>, $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$

& we "know" $m^{(j)}(0) = E(X^j)$

<u>Theorem</u> Let X be finite wp1. Then $c(t)$ is uniformly cts. If $E(|X|^k) < \infty$ then $E(X^k e^{itX})$ is uniformly cts (in t).

<u>Proof</u> Let $\epsilon > 0$. Then
$$|c(t_2) - c(t_1)|$$
$$= |E(e^{it_2 X} - e^{it_1 X})|$$

$$\leq E\left(\left|e^{it_2X}-e^{it_1X}\right|\right)$$

Now choose A so that

$$2P(|X|\geq A) \leq \epsilon/2 .$$

Now

$$E\left(\left|e^{it_2X}-e^{it_1X}\right|\right)$$

$$= E\Big(\overbrace{|\qquad\qquad|}^{\leq |t_2-t_1||X|} \overbrace{I(|X|<A)}^{\leq A}\Big)$$

$$+ E\Big(\underbrace{|\qquad\quad|}_{\leq 2} I(|X|\geq A)\Big)$$

$$\leq A|t_2-t_1| + 2P(|X|\geq A)$$

$$\leq A|t_2-t_1| + \epsilon/2$$

$$\therefore |t_2-t_1| \leq \frac{\epsilon}{2A} \Rightarrow |c(t_2)-c(t_1)| \leq \epsilon$$

qed

Consequence

① $E(|X|^k) < \infty \Rightarrow c^{(k)}(0)$ exists

(try to show for $k=1$ by def'n of a derivative)

$c^{(k)}(0)$ exists $\Longrightarrow E(|X|^k) < \infty$?

No unless k is even.

② $E(|X|^k) < \infty$

$$\Rightarrow c(t) = \sum_{j=0}^{k} \frac{(it)^j}{j!} E(X^j) + o(t^k)$$

$$\boxed{c^{(k)}(0) = i^k E(X^k)}$$

Some more background

real #'s

set of #'s — bounded above, bounded below } bounded

$\exists$ lub = sup

& glb = inf

sequences $a_n \to a$

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k \to a \right)$$

$\to$ has the Cauchy property in

mutual conv

That if $a_n - a_m \to 0$ as

$n + m \to \infty$ then $\exists$ an a st

$a_n \to a$

Types of convergence

1. $X_n \xrightarrow{ms} X$ (in mean square) if

$$E(X_n - X)^2 \to 0 \quad \text{as } n \to \infty$$

Fact $\xrightarrow{ms}$ has the Cauchy property

2. $X_n \xrightarrow{P} X$ (convergence in probability)

$\forall \epsilon > 0$

$$P(|X_n - X| \le \epsilon) \to 1 \text{ as } n \to \infty$$

Fact $\xrightarrow{P}$ has the Cauchy property

3. $X_n \to X$ if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$,

$\forall \omega \in \Omega$.

almost surely

4. $X_n \xrightarrow{wp1} X$ ($X_n \xrightarrow{as} X$) if $P(X_n \to X) = 1$

Theorem If $0 \le X_n$ & $X_n \uparrow X$ wp1 then $E(X_n) \to E(X)$.

$\boxed{\text{Monotone convergence theorem (MCT)}}$

Note $0 \le X_n$ can be replaced by assuming $E(|X_1|) < \infty$.

Theorem (DCT) If $X_n \xrightarrow{as} X$ & $|X_n| \le Y$ with $E(Y) < \infty$ then

$$E(X_n) \to E(X)$$

Note $X \overset{as}{=} X' \Rightarrow E(X) = E(X')$ which allows $X_n \uparrow X$ to be replaced by $X_n \uparrow X$, wp1. (Why?)

Another type of convergence

Def'n X_n converges to X weakly if $\forall$ bounded cts h we have

$$E(h(X_n)) \to E(h(X)) \quad (*)$$

Note — also called convergence in dist'n

— $X_n \xrightarrow{d} X$

separating class of h's (subset of all bd'd cts f'ns such that if $(*)$ holds for them it holds for all bd'd cts f'ns)

$$\underbrace{\{\sin tx, \cos tx \mid \forall t\}}_{\{e^{itx} \mid \forall t\}}$$

i.e. if $c_n(t) \to c(t)$, $\forall t$

then $X_n \xrightarrow{d} X$.

<u>Weak Law of Large Numbers (WLLN)</u>

Let $X_1, X_2, \ldots$ be iid with mean μ. Then

$$\bar{X} \xrightarrow{d} \mu$$

<u>Note</u> $Y_n \xrightarrow{d} c \Rightarrow Y_n \xrightarrow{P} c$ } problem

Proof: Let $c(t)$ be the cf of X_1. We know

$$c(t) = 1 + i\mu t + o(t) = e^{i\mu t + o(t)}$$

So

$$E\left(e^{it\bar{X}}\right) = E\left(e^{i\frac{t}{n}(X_1+\cdots+X_n)}\right)$$

$$= E\left(e^{i\frac{t}{n}X_1}\right)\cdots E\left(e^{i\frac{t}{n}X_n}\right)$$

$$= \left(c\left(\tfrac{t}{n}\right)\right)^n$$

$$= \left[e^{i\mu\frac{t}{n} + o\left(\frac{t}{n}\right)}\right]^n$$

$$= e^{i\mu t + n\,o\left(\frac{t}{n}\right)}$$

$$= e^{i\mu t + o\left(\frac{t}{n}\right)/\left(\frac{1}{n}\right)}$$

$$\to e^{i\mu t}$$

which is the cf of the constant μ.

qed

Central Limit Theorem

Let $X_1, X_2, \cdots$ be iid with mean μ & variance σ^2. Then

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

Proof The cf of X_1 is

$$c(t) = E\left(e^{itX_1}\right)$$

$$= e^{it\mu} E\left(e^{it\overbrace{(X_1-\mu)}^{\substack{\text{2nd moment}=\sigma^2 \\ \text{mean}=0}}}\right)$$

$$= e^{it\mu}\left(1 - \frac{\sigma^2 t^2}{2} + o(t^2)\right)$$

$$= e^{it\mu} e^{-\sigma^2 t^2/2} + o(t^2)$$

$$= e^{it\mu - \sigma^2 t^2/2 + o(t^2)}$$

$$E\left(e^{it \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}\right)$$

$$= E\left(e^{i\frac{t}{\sigma\sqrt{n}}[(X_1-\mu)+\cdots+(X_n-\mu)]}\right)$$

$$= \left(E\left(e^{i\frac{t}{\sigma\sqrt{n}}(X_1-\mu)}\right)\right)^n$$

$$= \left(e^{-\frac{\sigma^2 t^2}{2\sigma^2 n} + o\left(\frac{t^2}{\sigma^2 n}\right)}\right)^n$$

$$= e^{-t^2/2 + n\, o\left(\frac{t^2}{\sigma^2 n}\right)}$$

$$\to e^{-t^2/2}$$

qed

<u>Theorem (DCT)</u> Suppose $X_n \to X$ & $|X_n| \le W$ with $E(W) < \infty$. Then

$$E(X_n) \to E(X)$$

<u>Note</u> The conditions $X_n \to X$ & $|X_n| \le W$ can be replaced by $X_n \xrightarrow{as} X$ & $|X_n| \overset{as}{\le} W$

<u>Proof</u> Let $Z_n = |X_n - X| \le |X_n| + |X| \le 2W$

Then, setting $Y_n = \sup_{k \ge n} Z_k$, yields

$$0 \le Z_n \le Y_n \le 2W$$

Now $E(Y_1) < \infty$ & $Y_n \downarrow 0 \Rightarrow -Y_n \uparrow 0$ with $E(-Y_1)$ existing. Now use the MCT to get $E(Y_n) \to 0$. Since $0 \le Z_n \le Y_n$ we have $E(Z_n) \to 0$. Finally

$$|E(X_n) - E(X)| \le E|X_n - X| \to 0$$

so that $E(X_n) \to E(X)$

qed

$\xrightarrow{as}$

$X_n \xrightarrow{as} X$ if $P(X_n \to X) = 1$ (*)

$\{X_n \to X\} = \{\omega \mid X_n(\omega) \to X(\omega)\}$. Denote this event by D.

An equivalent def'n of $\xrightarrow{as}$ is (**)

$X_n \xrightarrow{as} X$ if $\forall \epsilon > 0$ $P(|X_m - X| \le \epsilon, \forall m \ge n) \to 1$

(ie $\forall \epsilon > 0$ $P(\sup_{m \ge n} |X_m - X| \le \epsilon) \to 1$)

Remark 1 $X_n \xrightarrow{as} X \Leftrightarrow |X_n - X| \xrightarrow{as} 0$, as $n \to \infty$

$\Leftrightarrow \sup_{m \ge n} |X_m - X| \xrightarrow{P} 0$, as $n \to \infty$

2 $X_n \xrightarrow{P} X \Leftrightarrow |X_n - X| \xrightarrow{P} 0$

From 1 & 2 we see that

$$X_n \xrightarrow{as} X \Rightarrow X_n \xrightarrow{P} X$$

$(*) \Leftrightarrow (**)$

Assume $(**)$. The events $\bigcap_{m=n}^{\infty} \{|X_m - X| \le \epsilon\}$ form an increasing sequence (in n) with limit

$$\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{|X_m - X| \le \epsilon\}$$

Now let $\epsilon_k \downarrow 0$ and set

$$D_k = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{|X_m - X| \le \epsilon_k\}$$

We have $P(D_k) = 1$, $\forall k$, and D_k forms a decreasing sequence in k. In fact

$$D_k \downarrow D$$

so that $P(D) = 1$ & hence $(*)$ holds.

Assume $(*)$. Then for D_k as above we have

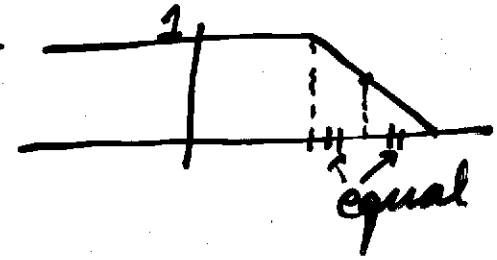
$$1 \geq P(D_k) \geq P(D) = 1$$

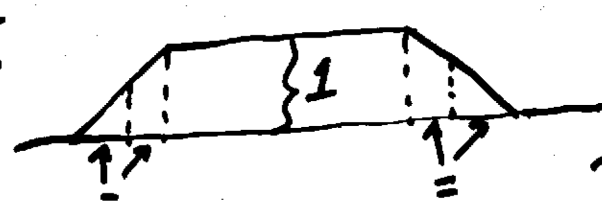
so that $P(D_k) = 1$, $\forall k$ & consequently $\forall \epsilon > 0$

$$P\left(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{|X_m - X| \le \epsilon\}\right) = 1$$ & so $(**)$ holds.

$$
\begin{tikzcd}
\xrightarrow{as} \arrow[dr, Rightarrow, "\text{clear}"'] & & \xrightarrow{ms} \arrow[dl, Rightarrow, "\text{Markov}"] \\
& \xrightarrow{P} \arrow[d, Rightarrow, "\text{tricky}"] & \\
& \xrightarrow{d} &
\end{tikzcd}
$$

<u>Proposition</u> $X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X$

A separating class $\left\{ \right.$  $\left. \right\}$

Another $\left\{ \right.$  $\left. \right\}$

Either of these can be used to show $\xrightarrow{d}$ $\iff F_n(x) \to F(x)$ at continuity points of F

<u>Proposition</u> $X_n \xrightarrow{d} c \implies X_n \xrightarrow{P} c$

<u>Proof</u> Let $\epsilon > 0$. Then

$$\begin{aligned} P(|X_n - c| > \epsilon) &= P(X_n < c-\epsilon) + P(X_n > c+\epsilon) \\ &\le P(X_n \le c-\epsilon) + [1 - P(X_n \le c+\epsilon)] \\ &\to 0 \end{aligned}$$

qed

<u>Little results</u>

$X_n \xrightarrow{d} X \implies \underset{\text{cts}}{g}(X_n) \xrightarrow{d} g(X)$ — true in the vector case

$X_n \xrightarrow{P} X \implies \underset{\text{cts}}{g}(X_n) \xrightarrow{P} g(X)$ — "

<u>Subsequence</u>

$X_1, X_2, \cdots$

$n_1 < n_2 < \cdots$ natural #'s

$X_{n_1}, X_{n_2}, \cdots$ is a subsequence

<u>Proposition</u> If $\forall \epsilon > 0 \quad \sum P(|Y_m| > \epsilon) < \infty$ then $Y_m \xrightarrow{as} 0$.

<u>Proof</u> Let $\epsilon > 0$. Then

$$P(|Y_m| \leq \epsilon\,;\ \forall m \geq N)$$

$$= 1 - P\left(\bigcup_{m=N}^{\infty} \{|Y_m| > \epsilon\}\right)$$

$$\overset{\text{Boole}}{\geq} 1 - \sum_{m=N}^{\infty} P(|Y_m| > \epsilon) \longrightarrow 1$$

qed

<u>Corollary 1</u> Let $\epsilon_m \downarrow 0$. Then $\sum P(|Y_m| > \epsilon_m) < \infty$ $\Rightarrow Y_m \xrightarrow{as} 0$

<u>Corollary 2</u> $X_m \xrightarrow{P} X \Rightarrow \exists\ X_{m_k} \xrightarrow{as} X$

<u>Pf of 2</u> Let $\epsilon_k \downarrow 0$. Choose k st

$$P(|X_{m_k} - X| > \epsilon_k) \leq c_k\,,$$

where $\sum c_k < \infty$. The result then follows from Corollary 1

<u>Corollary 3</u> $X_n \xrightarrow{P} X \iff$ every subsequence has a further subsequence converging wp1 to X

<u>Proof</u> Use contradiction.

<u>Recall</u> X_n converges mutually wp1 if $\forall \epsilon > 0 \quad P(|X_m - X_n| \le \epsilon, \forall m, n \ge N) \to 1$ as $N \to \infty$. This implies $\exists X$ st $X_n \xrightarrow{as} X$.

<u>Note</u> $X_n \xrightarrow{as} X$ if $P(|X_n - X| \le \epsilon, \forall n \ge N) \to 1$

<u>Proposition</u> Let $\epsilon_n > 0$, $\sum \epsilon_n < \infty$ &

$$\sum P(|X_{n+1} - X_n| > \epsilon_n) < \infty$$

Then $X_n \xrightarrow{as} X$

<u>Proof</u> Let $c_N = \sum_{n=N}^{\infty} \epsilon_n$. Then $c_N \downarrow 0$.
So, for any $\epsilon > 0$ $c_N \le \epsilon$ $\forall N$ large enough. Now for such N

$$P(|X_m - X_n| \le \epsilon, \forall n, m \ge N)$$

$$\geq P(|X_m - X_n| \leq c_N, \ \forall m,n \geq N)$$

$$\geq P(|X_{n+1} - X_n| \leq \epsilon_n, \ \forall n \geq N)$$

$$\geq 1 - \sum_{n=N}^{\infty} P(|X_{n+1} - X_n| > \epsilon_n) \to 1$$

Hence $\{X_n\}$ is mutually convergent wp1 so that $\exists\, X$ st $X_n \xrightarrow{as} X$

qed

<u>PDCT</u> $X_n \xrightarrow{P} X$, $|X_n| \leq W$ with $E(W) < \infty$. Then

$$E(X_n) \to E(X)$$

Pf: Use subsequences

<u>Note</u> In fact $E|X_n - X| \to 0$