Some solutions

#1. $A_n \uparrow A \Rightarrow A_1 \subset A_2 \subset \cdots$ and $I(A_n) \to I(A)$.

We must show $A = \cup A_k$. We first show $I(A_n) \to I(\cup A_k)$. To see this let $\omega \in \cup A_k$

$\Rightarrow \omega \in A_{k_0}, \omega \in A_{k_0+1}$, etc...

$\Rightarrow I(A_{k_0})(\omega) = I(A_{k_0+1})(\omega) = \cdots = 1$

Since $I(\cup A_k)(\omega) = 1$ we have

$$\lim_{n\to\infty} I(A_n)(\omega) = I(\cup A_k)(\omega), \quad \forall \omega \in \cup A_k$$

On the other hand if $\omega \notin \cup A_k$ (ie $\omega \in (\cup A_k)^c$) then $I(\cup A_k)(\omega) = 0$ and

$$I(A_1)(\omega) = I(A_2)(\omega) = \cdots = 0,$$

so that

$$\lim_{n\to\infty} I(A_n)(\omega) = I(\cup A_k)(\omega), \quad \forall \omega \notin \cup A_k$$

$\therefore$
$$\lim_{n\to\infty} I(A_n)(\omega) = I(\cup A_k)(\omega), \quad \forall \omega \in \Omega$$

$$\Rightarrow I(A_n) \to I(\cup A_k)$$

Now suppose $I(A_n) \to I(A)$ and $I(A_n) \to I(A')$.
Then $I(A) = I(A')$ and this can only happen if $A = A'$. $\therefore$ limits of sequences of events are unique and hence

$$A = \bigcup A_k$$

#2. Let $\omega \in \left(\bigcup A_k\right)^c \Leftrightarrow \omega \notin \bigcup A_k$

$\Leftrightarrow \omega \in \bigcap A_k^c$ $\therefore$

$$\left(\bigcup A_k\right)^c = \bigcap A_k^c$$

Now consider the eq'n

$$\left(\bigcap A_k\right)^c = \bigcup A_k^c$$

This is just

$$\bigcup A_k^c = \left(\bigcap A_k\right)^c \quad (*)$$

Let $B_k = A_k^c$. Then $(*)$ is just

$$\bigcup B_k = \left(\bigcap B_k^c\right)^c$$

$\Leftrightarrow \left(\bigcup B_k\right)^c = \bigcap B_k^c$ which we first showed.

3 - done in class

4 - Let $\omega \in A \cup B$. Then $I(A \cup B)(\omega) = 1$ while $I(A)(\omega) + I(B)(\omega) - I(AB)(\omega) = 1$ depending on wheter ω is only in A or only in B or in both.

If $\omega \notin A \cup B$ then $I(A \cup B)(\omega) = 0$ as is $I(A)(\omega)$, $I(B)(\omega)$ & $I(AB)(\omega)$.

So $I(A \cup B)(\omega) = I(A)(\omega) + I(B)(\omega) - I(AB)(\omega)$, $\forall \omega \in \Omega$

$\Rightarrow I(A \cup B) = I(A) + I(B) - I(AB)$

5 & challenge done in class

Text

p16 #1 Assume Axioms 1 & 4. Then if $a \leq X \leq b$ then $X - a$ & $b - X \geq 0$. Now use Axioms 1, 2, 3 to get $E(X) \geq a$ & $b \geq E(X)$. That is

$$a \leq E(X) \leq b$$

Assume $a \leq X \leq b \Rightarrow a \leq E(X) \leq b$ for constants a, b (as well as Axioms 2 & 3). We need to verify Axioms 1 & 4.

Let $X \geq 0$. Set $X_n = X I(X \leq n)$. Then $X_n \uparrow X$ & so $E(X_n) \to E(X)$ by Ax 5. But $0 \leq X_n \leq n$ & so $0 \leq E(X_n) \leq n$. Hence $0 \leq E(X)$.

To show $E(1) = 1$ just take $a = b = 1$.

p16 # 3 — done in class

p16 # 5 — We have $0 \leq |X_n - X| \leq Y_n$ & so

$$0 \leq E(|X_n - X|) \leq E(Y_n)$$

Since $E(Y_n) \to 0$ we obtain

$$E(|X_n - X|) \to 0 \quad \text{(this is } \mathcal{L}_1 \text{ convergence)}$$

The result then follows from

$$|E(X_n) - E(X)| = |E(X_n - X)| \leq E(|X_n - X|)$$

p36 #4

Let $g(x) = 1 \quad , \quad x \geq 0$
$\qquad = 0 \quad , \quad ow$

Then $g(x) \leq \dfrac{H(x)}{H(a)} \quad , \quad \forall x$

$$\Rightarrow P(X \geq a) = E[g(X)] \leq \frac{E[H(X)]}{H(a)}$$

p36 #11 – done in class

p36 #14

$$P(|\bar{Y}_n - \mu| > \epsilon) = P\left((\bar{Y}_n - \mu)^2 > \epsilon^2\right)$$

$$\leq \frac{E[(\bar{Y}_n - \mu)^2]}{\epsilon^2} \qquad \text{(Markov)}$$

$$\longrightarrow 0$$