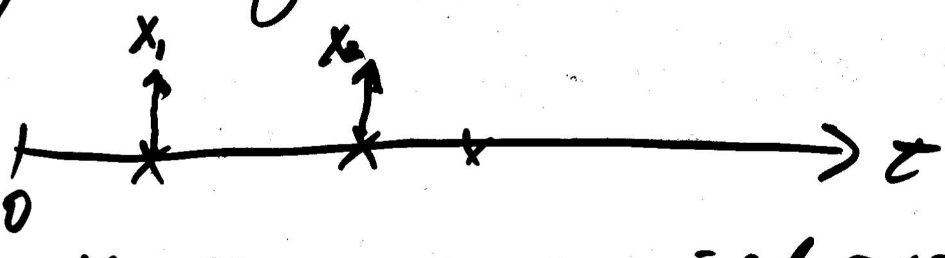
Problems

1. Let $\{X_t : t = 0, 1, \ldots\}$ be a Galton Watson branching process with offspring pgf $G(s) = \frac{1}{2}s^2 + \frac{1}{4}s + \frac{1}{4}$. Take $X_0 = 1$. Find the probability of ultimate extinction.

2. Consider a Poisson process of rate 2 & the (marked) compound process



where $X_1, X_2, \ldots$ are iid exponential$(\frac{1}{4})$. Calculate $\mathrm{Var}\left(\sum^{N(t)} X_i\right)$.

3. Let $\{N(t) \mid t \geq 0\}$ be a nonhomogeneous Poisson process of rate $\lambda(t)$. Show $N(t) \sim \mathrm{Poisson}\left(\int_0^t \lambda(u)\,du\right)$

Solutions

#1 Let ρ denote the probability of ultimate extinction. Then ρ satisfies

$$G(\rho) = \rho \quad , \quad 0 \le \rho \le 1 .$$

This is just

$$\frac{1}{2}\rho^2 - \frac{3}{4}\rho + \frac{1}{4} = 0 \quad , 0 \le \rho \le 1$$

which has roots $\rho = \frac{1}{2}$, $\rho = 1$.
The smaller of the roots is ρ so that $\rho = \frac{1}{2}$

#2 $N(t) \sim Poisson(2t)$. Set

$$S = \sum_{i=0}^{N(t)} X_i \quad , \quad X_0 = 0$$

Then $E[S \mid N(t)] = 4N(t)$ while

$$Var[S \mid N(t)] = 16\,N(t) .$$

(4 & 16 are the mean & variance of an exponential($\frac{1}{4}$))

$$\therefore\ Var(S) = E[Var(S \mid N(t))] + Var(E(S \mid N(t)))$$

$$= 16 \times 2t + 16 \times 2t = 64t$$