Basics, extensions from STA347

Note Covers material up until Feb 11

1. Let $\{N(t) \mid t \geq 0\}$ be a renewal process with interarrival times $X_1, X_2, \ldots$ which are iid having mean μ and variance σ^2.

(a) Show $\dfrac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} \xrightarrow{d} N(0,1)$

(b) Show $E(N^k(t)) < \infty$, $\forall t \geq 0$ & $k \in \mathbb{N}$.

(c) If $\mu = \infty$ show
$$\frac{E(N(t))}{t} \to 0$$

(d) If X_1 has a possibly different dist'n than $X_2, X_3, \ldots$ show

$$\frac{N(t)}{t} \xrightarrow{as} \frac{1}{\mu} \quad \& \quad \frac{m(t)}{t} \to \frac{1}{\mu}$$

(here $m(t) = E(N(t))$ & μ is finite)

2.(a) Let $X_1, X_2, \dots$ be iid with $p = P(X_1 = 1)$, $q = P(X_1 = -1)$, $p+q=1$, $0<p<1$. Set $S_0 = 0$, $S_n = X_1 + \dots + X_n$ for $n = 1, 2, \dots$. Show $P(S_n = 0 \text{ io}) = 1 \Leftrightarrow p = q$.

(b) Let $X_1, X_2, \dots$ be rv's. Show $\exists$ constants $c_n > 0$ such that

$$c_n X_n \xrightarrow{as} 0$$

(c) Give an example where $Y_n \xrightarrow{as} 0$ but $E(Y_n) \not\to 0$

(d) Let $X_1, X_2, \dots$ be iid integer valued rv's with $E(X_1) = 0$. Set $S_0 = 0$ & $S_n = X_1 + \dots + X_n$. It can (& will) be shown $P(S_n = 0 \text{ io}) = 1$. Assume $P(X_n = 1) > 0$ and let $X_1^*, X_2^*, \dots$ be iid X_1 and independent of the X's. For any $k \in \mathbb{Z}$ show $\exists$ n, m such that

$$P(S_n - S_m^* = k) > 0$$

3(a) Let $X_n \xrightarrow{ms} X$ and $X_n \xrightarrow{P} X'$. Show $X \overset{wp1}{=} X'$

(b) Suppose $|X_n| \leq W$ & $E(W) < \infty$. If $X_n \xrightarrow{P} X$ show $E(|X_n - X|) \to 0$.

(c) Suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} c$. Show $X_n Y_n \xrightarrow{d} cX$. Further, if $c \neq 0$ show $\dfrac{X_n}{Y_n} \xrightarrow{d} \dfrac{X}{c}$

(d)(i) Let $\mathcal{B}$ denote the Borel σ-field on $\mathbb{R}$. Let $\mathcal{C}$ be any collection of subsets of $\mathbb{R}$ such that $\sigma(\mathcal{C}) = \mathcal{B}$. Now take a probability space $(\Omega, \mathcal{F}, P)$ and a function $X : \Omega \to \mathbb{R}$. Show X is measurable wrt $\mathcal{F}$ & $\mathcal{B}$ if $X^{-1}(\mathcal{C}) \subset \mathcal{F}$

(ii) Show that any open set in $\mathbb{R}$ is a countable union of disjoint open intervals.

4. Let $\{X_t : t = 0, 1, 2, \cdots\}$ be a branching process with $X_0 = 1$, $\left\{\begin{array}{l} P(X_1 = 0) > 0, \\ P(X_1 > 1) > 0 \end{array}\right.$

$\mu = E(X_1)$ & $G(s) = E(s^{X_1})$

(a) Let $\rho_0 = \lim_{n \to \infty} P(X_n = 0)$. Show that ρ_0 is the smallest positive number satisfying

$$\rho_0 = \sum_{j=0}^{\infty} \rho_0^j P(X_1 = j)$$

(b) For $m \le n$ show

$$E(X_m X_n) = \mu^{n-m} E(X_m^2)$$

(c) If $G_n(s) = E(s^{X_n})$ show $G_n(s) = G_{n-1}[G(s)]$

(d) If $G(s) = 1 - \alpha(1-s)^\beta$ where $0 < \alpha, \beta < 1$ find the pgf of X_n.

5(a) Let $X_0, X_1, \cdots$ be such that $X_n \sim N(0,1)$, any finite collection of the X's is multivariate normal and $\mathrm{cov}(X_n, X_m) = \rho^{|n-m|}$ for some $\rho \in [0,1)$. Show

$$f(x_0, x_1, \cdots, x_n) = f(x_0)\, f(x_1|x_0) \cdots f(x_n|x_{n-1})$$

(b) Points are randomly distributed in 3-d in such a way that the # of points in a region of volume V is Poisson(2·V). Let Y = distance from $\underset{\sim}{0}$ to the nearest point. Calculate E(Y).

(c) Suppose $\underset{\sim}{X}_n \xrightarrow{P} \underset{\sim}{X}$ & $g: \mathbb{R}^k \longrightarrow \mathbb{R}$ is continuous. Show $g(\underset{\sim}{X}_n) \xrightarrow{P} g(\underset{\sim}{X})$.

(d) Let $X_1, X_2, \cdots$ be iid with $E(X_i) = 0$. Assume $E(X_1^4) < \infty$. Prove $\bar{X} \xrightarrow{a.s.} 0$.

$\sigma(\underset{\sim}{X}_m)$ will denote the σ-field $\underset{\sim}{X}^{-1}(\mathcal{B}_m)$. For any sequence of rv's $X_1, X_2, \cdots$ $\sigma(X_1, X_2, \cdots)$ will be the smallest σ-field generated by any finite number of the X's. It can be shown that for $A \in \sigma(X_1, X_2, \cdots)$ and $\epsilon > 0$ $\exists$ $A_n \in \sigma(X_1, \cdots, X_n)$ such that $P(A \Delta A_n) \leq \epsilon$, where $A \Delta A_n = AA_n^c \cup A^c A_n$.

6(a) Let $X_1, X_2, \cdots$ be independent rv's and let $A \in \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \cdots)$. Show $P(A)$ is either 0 or 1.

(b) Let $X_1, X_2, \cdots$ be ≥ 0 independent rv's. Can you find such X's so that

$$P\left(\sum_{k=1}^{\infty} X_k < \infty\right) = 3/4 \ ?$$

(c) Let $x_1, \cdots, x_n \geq 0$. Show $\bar{x} \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$.

Hint: Use Jensen's inequality.

(d) Let $X \in \{1, 2, \cdots\}$ have pgf $G(s)$. Suppose $\mu = E(X)$. Show $D_*(s) = \frac{1 - G(s)}{\mu(1-s)}$ can be written as $\sum_{k \geq 0} a_k s^k$ where $a_k \geq 0$ and $\sum a_k = 1$

7. Let $\{X_t : t=0,1,...\}$ be a Galton Watson branching process with $X_0=1$ and *geometric(p)* offspring distribution - $0<p<1$. Let T be the first time the population becomes extinct. Obtain $P(T=k)$ and determine which values of p lead to $E(T) < \infty$.

8. Let $\{F_t$, t in T$\}$ be σ-fields of events. Show $\bigcap_{t\in T} F_t$ is also a σ-field. Give an example of 2 σ-fields F_1, F_2 where $F_1 \cup F_2$ is not a σ-field .

9. For a < b in $\Re$ show $\sigma(\{(a,b)\}) = \sigma(\{[a,b]\}) = \sigma(\{\text{open subsets}\})$.

10. Show $X_n \xrightarrow{ms} X \Leftrightarrow X_n - X_m \xrightarrow{ms} 0$, as $n,m \to \infty$.

11. Show $X_n \xrightarrow{p} 0 \Leftrightarrow E\left(\frac{|X_n|}{1+|X_n|}\right) \to 0$.

12. For X, Y in L_2 define $\langle X,Y\rangle = E(XY)$ and $||X|| = \sqrt{\langle X,X\rangle}$. Verify

(i) $\langle aX+bY,Z\rangle = a\langle X,Z\rangle+b\langle Y,Z\rangle$

(ii) $||X+Y||^2+||X-Y||^2 = 2||X||^2+2||Y||^2$

(iii) if $i\neq j \Rightarrow \langle X_i, X_j\rangle \neq 0$ then

$$||\sum_{i=1}^{n} X_i||^2 = \sum_{i=1}^{n} ||X_i||^2$$

(iv) IIXII=0 implies X=0 in mean square and wp1

* We are using a modified geometric so that $X_1 + 1 \sim$ geometric(p)

Basics

1. Show $E|X| < \infty$ iff $\sum_{k=0}^{\infty} P(|X|>k) < \infty$.

2. Let $\{X_n\}$ & $\{Y_n\}$ be such that $P(X_n \neq Y_n \text{ io}) = 0$. Show
$$\frac{1}{n} \sum_{k=1}^{n} (Y_k - X_k) \xrightarrow{as} 0$$

3. Let $X_1, X_2, \ldots$ be iid X where $E|X|<\infty$. Define the <u>truncated</u> rv's $\tilde{X}_n$ by $\tilde{X}_n = X_n$ for $|X_n| < n$ and 0 otherwise. Show $P(\tilde{X}_n \neq Y_n \text{ io}) = 0$.

4. Suppose $\sum_{k=1}^{\infty} a_k = a$ while $b_n \uparrow +\infty$. Show $\frac{1}{b_n} \sum_{k=1}^{n} b_k a_k \to 0$.

5. Let $X_1, X_2, \ldots$ be independent and T a tail rv. Show $T \overset{as}{=}$ constant.

6. We will show for independent $X_1, X_2, \ldots$
$$\sum_{k=1}^{n} X_k \xrightarrow{ms} \;\Rightarrow\; \sum_{k=1}^{n} X_k \xrightarrow{as} \quad \left(\Leftrightarrow \sum_{k=1}^{n} X_k \xrightarrow{P} \;!!\right)$$
Now, suppose $X_1, X_2, \ldots$ are iid with mean 0. Prove
$$\bar{X} \xrightarrow{as} 0 \qquad \text{(SLLN)}$$

<u>Hint:</u> Truncate & show $\sum_{k=1}^{\infty} \frac{E\tilde{X}_k^2}{k^2} < \infty$. You may need to interchange an order of summation in a double sum (can be done as all terms will be ≥ 0)

7(i) Let $W_n = X_n + Y_n$ & suppose $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{ms} 0$. Show $W_n \xrightarrow{d} X$.

(ii) $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} c \Rightarrow X_n + Y_n \xrightarrow{d} X + c$, $X_n Y_n \xrightarrow{d} cX$, $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}$ $(c \neq 0)$

Markov processes, conditioning

1. Let $\{X_0, X_1, \cdots\}$ be Markov. For $t_0 < t_1 < t_2 \cdots$ a subsequence of Z^+ show $\{X_{t_i}\}$ is also Markov.

2. Let $X_1, X_2, \cdots$ be iid & let $S_n = X_1 + \cdots + X_n$. Show $$E(X_1 \mid S_n, S_{n+1}, \cdots) = \frac{S_n}{n}$$

3. Let $\{X_t \mid t = 0, 1, \cdots\}$ be such that $X_t \mid X_0$ are independent. Show that this process need not be Markov, but that $\{\underset{\sim}{Y}_t \mid t = 0, 1, 2, \cdots\}$ with $\underset{\sim}{Y}_t = \begin{pmatrix} X_0 \\ X_t \end{pmatrix}$ is.

4. Suppose $\{X_t \mid t = 0, 1, \cdots\}$ is a finite MC which is ergodic (has a limiting dist'n not dependent on the initial dist'n). Show that X_r & X_s become independent as $|r - s| \to \infty$.

5. For a simple random walk on the integers (steps ± 1 with probabilities p & $q = 1 - p$) obtain $P_{ij}(n)$.

6. Let $\{X_t \mid t = 0, 1, \cdots\}$ be a MC and N a stopping time. Show $X_{N+K} \mid \{X_t, t \le T\} = X_{N+K} \mid X_T$ $\forall\, K \geq 0$ (an integer of course).

7. Let $\{X_t \mid t \in Z^+\}$ be a MC containing an absorbing state (once entered you never leave) accessible from all other states. Show that these other states must be transient.

8. Let $\{X_t \mid t \in Z\}$ be a MC which is aperiodic, irreducible and positive recurrent. Let $\underline{\Pi}$ denote its stationary distribution and suppose $X_t \sim \underline{\Pi}$, $\forall t$. Call $\{X_t\}$ time-reversible if $X_{t+1} \mid X_t \overset{d}{=} Y_{t+1} \mid Y_t$, where $Y_t = X_{-t}$. Show this to be the case iff

$$\pi_i p_{ij} = \pi_j p_{ji}, \quad \forall \text{ states } i, j$$

9. N particles move independently between two regions of space. Each particle has a probability of 1/2 of moving to a different region over a unit of time. Let X_t be the # of particles in the first region at time t. State the transition matrix and evaluate the stationary distribution.

Hint: It's reasonable to assume that the process is time-reversible.

10 (a) Suppose a finite (state space) MC has a doubly stochastic transition matrix (rows + columns add to 1). Show that all states are positive recurrent. If the chain is also irreducible and aperiodic show it has a uniform stationary distribution $\underline{\Pi}$ with $p_{ij}(n) \to \pi_j$ (= 1/# of states).

(b) Show irreducible + doubly stochastic ⇒ all states are transient or null recurrent (in an ∞ state space).

renewal processes

1. Verify $U(s) = \dfrac{D(s)}{1-G(s)}$

2. Let $\{N(t) \mid t \geq 0\}$ be a renewal process with interarrival df F. Show
$$P(X_{N(t)+1} \geq x) \geq \bar{F}(x) = 1 - F(x)$$
and evaluate the LHS for a Poisson process of rate λ.

<u>Note</u> The interarrival times $X_1, X_2, \ldots$ are iid with df F

3. Verify the renewal equation
$$\underbrace{m(t)}_{E[N(t)]} = \underbrace{F(t)}_{\text{interarrival df}} + \underbrace{\int_0^t m(t-x)\, dF(x)}_{\text{an expectation}}$$

4. Let $U_1, U_2, \ldots$ be iid uniform$((0,1))$ & let N be the smallest n such that $U_1 + \cdots + U_n \geq 1$. Show $E(N) = e$.

5. Let $X_1, X_2, \ldots$ be iid with mean μ & $N_1, N_2, \ldots$ iid stopping times for the X's. Assume $E(N_1) < \infty$.
Let $S_1 = X_1 + \cdots + X_{N_1}$, $S_2 = X_{N_1+1} + \cdots + X_{N_1+N_2}$, etc ...

(a) Compute $\lim_{m\to\infty} \dfrac{S_1 + \cdots + S_m}{N_1 + \cdots + N_m}$

(b) Derive another expression for (a) via $\dfrac{S_1 + \cdots + S_m}{N_1 + \cdots + N_m} = \left(\dfrac{S_1 + \cdots + S_m}{m}\right)\left(\dfrac{m}{N_1 + \cdots + N_m}\right)$

(c) Use (a) & (b) to obtain Wald's Eq'n.