STA 447

Assignment 2

Basics

1. Show $E|X| < \infty$ iff $\sum_{k=0}^{\infty} P(|X| > k) < \infty$.

2. Let $\{X_n\}$ & $\{Y_n\}$ be such that $P(X_n \neq Y_n \text{ io}) = 0$. Show
$$\frac{1}{n} \sum_{k=1}^{n} (Y_k - X_k) \xrightarrow{as} 0$$

3. Let $X_1, X_2, \ldots$ be iid X where $E|X| < \infty$. Define the <u>truncated</u> rv's $\tilde{X}_n$ by $\tilde{X}_n = X_n$ for $|X_n| < n$ and 0 otherwise. Show $P(X_n \neq Y_n \text{ io}) = 0$.

4. Suppose $\sum_{k=1}^{\infty} a_k = a$ while $b_n \uparrow +\infty$. Show $\frac{1}{b_n} \sum_{k=1}^{n} b_k a_k \to 0$.

5. Let $X_1, X_2, \ldots$ be independent and T a tail rv. Show $T \overset{as}{=}$ constant.

6. We will show for independent $X_1, X_2, \ldots$
$$\sum_{k=1}^{n} X_k \xrightarrow{ms} \Rightarrow \sum_{k=1}^{n} X_k \xrightarrow{as} \left(\Leftrightarrow \sum_{k=1}^{n} X_k \xrightarrow{P} \; !!\right)$$
Now, suppose $X_1, X_2, \ldots$ are iid with mean 0. Prove
$$\bar{X} \xrightarrow{as} 0 \qquad \text{(SLLN)}$$
<u>Hint:</u> Truncate & show $\sum_{k=1}^{\infty} \frac{E\tilde{X}_k^2}{k^2} < \infty$. You may need to interchange an order of summation in a double sum (can be done as all terms will be ≥ 0)

7. (i) Let $W_n = X_n + Y_n$ & suppose $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{ms} 0$. Show $W_n \xrightarrow{d} X$.

(ii) $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} c \Rightarrow X_n + Y_n \xrightarrow{d} X + c$, $X_n Y_n \xrightarrow{d} cX$, $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}$ $(c \neq 0)$

Markov processes, conditioning

1. Let $\{X_0, X_1, \cdots\}$ be Markov. For $t_0 < t_1 < t_2 \cdots$ a subsequence of Z^+ show $\{X_{t_i}\}$ is also Markov.

2. Let $X_1, X_2, \cdots$ be iid & let $S_n = X_1 + \cdots + X_n$.
Show $$E(X_1 \mid S_n, S_{n+1}, \cdots) = \frac{S_n}{n}$$

3. Let $\{X_t \mid t = 0, 1, \cdots\}$ be such that $X_t \mid X_0$ are independent. Show that this process need not be Markov, but that $\{\underset{\sim}{Y}_t \mid t = 0, 1, 2, \cdots\}$ with $\underset{\sim}{Y}_t = \begin{pmatrix} X_0 \\ X_t \end{pmatrix}$ is.

4. Suppose $\{X_t \mid t = 0, 1, \cdots\}$ is a finite MC which is ergodic (has a limiting dist'n not dependent on the initial dist'n). Show that X_r & X_s become independent as $|r - s| \to \infty$.

5. For a simple random walk on the integers (steps ± 1 with probabilities p & $q = 1 - p$) obtain $p_{ij}(n)$.

6. Let $\{X_t \mid t = 0, 1, \cdots\}$ be a MC and N a stopping time. Show $X_{N+k} \mid \{X_t, t \leq T\} = X_{N+k} \mid X_T$ $\forall\ k \geq 0$ (an integer of course).

7. Let $\{X_t \mid t \in Z^+\}$ be a MC containing an <u>absorbing state</u> (once entered you never leave) accessible from all other states. Show that these other states must be transient.

8. Let $\{X_t \mid t \in Z\}$ be a MC which is aperiodic, irreducible and positive recurrent. Let $\underline{\Pi}$ denote its stationary distribution and suppose $X_t \sim \underline{\Pi}$, $\forall t$. Call $\{X_t\}$ <u>time-reversible</u> if $X_{t+1} \mid X_t \overset{d}{=} Y_{t+1} \mid Y_t$, where $Y_t = X_{-t}$. Show this to be the case iff

$$\pi_i p_{ij} = \pi_j p_{ji} \quad , \forall \text{ states } i, j$$

9. N particles move independently between two regions of space. Each particle has a probability of $1/2$ of moving to a different region over a unit of time. Let X_t be the # of particles in the first region at time t. State the transition matrix and evaluate the stationary distribution.

<u>Hint:</u> It's reasonable to assume that the process is time-reversible.

10 (a) Suppose a finite (state space) MC has a doubly stochastic transition matrix (rows + columns add to 1). Show that all states are positive recurrent. If the chain is also irreducible and aperiodic show it has a uniform stationary distribution $\underline{\Pi}$ with $p_{ij}(n) \to \pi_j$ $(= 1/\#\text{ of states})$.

(b) Show irreducible + doubly stochastic $\Rightarrow$ all states are transient or null recurrent (in an ∞ state space).

renewal processes

1. Verify $U(s) = \dfrac{D(s)}{1-G(s)}$

2. Let $\{N(t) \mid t \geq 0\}$ be a renewal process with interarrival df F. Show
$$P(X_{N(t)+1} \geq x) \geq \bar{F}(x) = 1 - F(x)$$
and evaluate the LHS for a Poisson process of rate λ.
Note The interarrival times $X_1, X_2, \dots$ are iid with df F

3. Verify the renewal equation
$$\underset{\underset{E[N(t)]}{\shortparallel}}{m(t)} = \underset{\underset{\text{interarrival df}}{\downarrow}}{F(t)} + \underbrace{\int_0^t m(t-x)\, dF(x)}_{\text{an expectation}}$$

4. Let $U_1, U_2, \dots$ be iid uniform$((0,1))$ & let N be the smallest n such that $U_1 + \dots + U_n \geq 1$.
Show $E(N) = e$.

5. Let $X_1, X_2, \dots$ be iid with mean μ & $N_1, N_2, \dots$ iid stopping times for the X's. Assume $E(N_1) < \infty$.
Let $S_1 = \underline{X_1 + \dots + X_{N_1}}$, $S_2 = X_{N_1+1} + \dots + X_{N_1+N_2}$, etc ...

(a) Compute $\displaystyle\lim_{m\to\infty} \frac{S_1 + \dots + S_m}{N_1 + \dots + N_m}$

(b) Derive another expression for (a) via $\dfrac{S_1 + \dots + S_m}{N_1 + \dots + N_m} = \left(\dfrac{S_1 + \dots + S_m}{m}\right)\left(\dfrac{m}{N_1 + \dots + N_m}\right)$

(c) Use (a) & (b) to obtain Wald's Eq'n.