Problem Set # 3

1. (a) Let $Y \geq 0$. Show $E(Y) < \infty$ iff $\forall \epsilon > 0 \;\exists\; \delta > 0 \ni$ $P(A) < \delta \Rightarrow E(Y I_A) \leq \epsilon$.

(b) For $Y \geq 0$ show $E(Y) < \infty \Leftrightarrow E[Y I(Y > y)] \to 0$ as $y \to \infty$.

2. Let X have df F. For $0 < p < 1$ define the pth quantile as $\inf_x \{x \mid F(x) \geq p\}$. Denote it by $F^{-1}(p)$. Show
(i) $F^{-1}[F(x)] \leq x$ (ii) $F[F^{-1}(p)] \geq p$ (iii) $F(x) \geq p$ iff $x \geq F^{-1}(p)$

3. Let $X_1, X_2, \cdots$ be independent rv's with 0 means + set $S_m = \sum_{i=1}^{m} X_i$. For $c > 0$ show $P\left(\max_{1 \leq k \leq m} |S_k| > c\right) \leq \frac{Var(S_m)}{c^2}$.
Now use this to show $\sum_{k=1}^{\infty} E(X_k^2) < \infty \Rightarrow \sum_{k=1}^{\infty} X_k$ converges wp1 and combine this with previous results to prove the SLLN.

4(a) Let $\{N(t) \mid t \geq 0\}$ be a renewal process with iid interarrivals X_m. Show $N(t) \overset{as}{<} \infty$, $\forall t \Leftrightarrow E(X_1) > 0$.

(b) Show $\{N(t)/t \mid t \geq 1\}$ is uniformly integrable.

(c) Denote the df of X_m by F + the time to the mth renewal by S_m + its df by F_m. Show
$$P(S_{N(t)} \leq s) = \bar{F}(t) + \sum_{k=1}^{\infty} E_k[\bar{F}(t-y)]$$

<u>Note</u> By $E_k[g(y)]$ we mean $E[g(S_k)]$.

(d) Obtain the renewal function $m(t)$ when $X_m \sim \text{gamma}(2, \lambda)$

5. Let $\{X(t) \mid t \geq 0\}$ be a simple B+D process with $X(0) = i$. Obtain the pgf of $X(t)$ and calculate $\lim_{t \to \infty} P[X(t)=0 \mid X(0)=i]$

<u>Hint</u> The pgf of $X(t)$ must be of the form $(\;)^i$. Now use the backward equations.

6(a) For a Yule process (as in #5 but $\lambda_n = n\lambda$ & $\mu_n = 0$) calculate $p_{ij}(t)$.

(b) Take a Yule process with $X(0) = 1$ & suppose it is known $X(t) = n+1$. Show that the n births are at times corresponding to the order statistics of a sample of size n from the pdf

$$f(x) = \frac{\lambda e^{-\lambda(t-x)}}{1-e^{-\lambda t}}, \quad 0 < x < t$$

7. Let $\{X(t) \mid t \geq 0\}$ be a B+D process with rates $\lambda_n = \lambda$, $\mu_n = n\mu$ + $X(0) = i$. Show that the pgf of $X(t)$ is given by

$$\left(1+(z-1)e^{-\mu t}\right)^i \exp\left[\frac{\lambda}{\mu}(z-1)(1-e^{-\mu t})\right]$$

What is the limiting dist'n of $X(t)$ as $t \to \infty$?

8. Let $\{X(t) \mid t \geq 0\}$ be a birth process with birth rates λ_n satisfying $\sum \frac{1}{\lambda_n} < \infty$. Show $\exists\ t_0$ st $\sum_j P(X(t_0)=j \mid X(0)=i) < 1$.