Problem Set #3 ct'd

1(a) Let $\{Y_t\}$ with $Y_t \geq 0$ be ui. Show $\sup_t E(Y_t) < \infty$.

(b) Let $\{X_n\}$ be ui & suppose $X_n \xrightarrow{P} X$. Show $X_n \xrightarrow{L_1} X$ & $E(X_n) \to E(X)$ & $E(|X_n|) \to E|X|$

(c) Let $X_n \in L_1$ & $X_n \xrightarrow{L_1} X$. Show $\{X_n\}$ is ui.

(d) Let $X_n \in L_1$, $X_n \xrightarrow{P} X$ & $E|X_n| \to E|X|$. Show $\{X_n\}$ is ui.

2(a) Let $X_1, X_2, \cdots$ satisfy $E(X_n / \underset{\sim}{X}_{n-1}) = 0$, $\forall n \geq 1$ where $\underset{\sim}{X}_k = (X_1, \cdots, X_k)'$ & $X_0 = 0$ for convenience. Set $S_n = X_1 + \cdots + X_n$. Show $E(S_n / \underset{\sim}{S}_m) = S_m$ for $1 \leq m < n$ & $E(S_n) = 0$, $\forall n$ (set $S_0 = 0$).

(b) For the situation in 2(a) show $Var(S_n) = \sum_{k=1}^{n} Var(X_k)$

(c) If $\{X_n\}$ and $\{S_n\}$ are as in 2(a) & $\{Y_n\}$ is such that $\underset{\sim}{X}_n = g(\underset{\sim}{Y}_n)$ & $E(S_n / \underset{\sim}{Y}_m) = S_m$ for $1 \leq m < n$ show $E(S_n / \underset{\sim}{S}_m) = S_m$, $1 \leq m < n$ while $E(X_n / \underset{\sim}{X}_{n-1}) = E(X_n / \underset{\sim}{Y}_{n-1}) = 0$, $\forall n \geq 1$ ($X_0 = Y_0 = 0$).

Remark – $\{S_n\}$ in 2(a) is a (zero mean) martingale.
– $\{S_n\}$ in 2(c) is a martingale wrt $\{Y_n\}$
– $\{c + S_n\}$ is a martingale with $E(c + S_n) = c$

3(a) Let $W_1, W_2, \ldots$ have pdf $f(\underset{\sim}{w}_n|\theta)$ where $\in \mathbb{R}$. Show, assuming reasonable conditions, that

$$\left\{\frac{\partial \log L_n(\theta)}{\partial \theta}\right\}$$

is a martingale. Here, $L_n(\theta) \doteq f(\underset{\sim}{W}_n|\theta)$ is the likelihood f'n. In this context it is usual to avoid the upper/lowercase notation for rv's.

(b) Let $X_1, X_2, \ldots$ be iid with $E(X_i) = 0$, $\forall i$. Show $\{S_n\}$ is a martingale

(c) Let $\{Z_n, n \geq 0\}$ be a branching process with $Z_0 = 1$, offspring mean μ & probability of ultimate extinction ρ. Show $\{Z_n/E(Z_n)\}$ and $\{\rho^{Z_n}\}$ are both martingales wrt $\{Z_n\}$. Now add immigration in each generation with mean m. Show

$$\left\{\frac{1}{\mu^n}\left[Z_n - m\left(\frac{1-\mu^n}{1-\mu}\right)\right]\right\}$$

is also a martingale (assume $\mu \neq 1$).

(d) Let $\{Z_n\}$ be a MC & h : State space $\to \mathbb{R}$ satisfies $P\underset{\sim}{h} = \underset{\sim}{h}$, where P is the transition matrix. Show $\{h(Z_n)\}$ is a martingale.