Suggested Problems # 4

1. Let $\{S_n\}$ be a martingale with mean S_0. If $T \geq 1$ is a stopping time show (a) $T_n = T \wedge n$ is also a stopping time and (b) $\{S_{T_n}\}$ is a martingale.

2. Let $\{S_n\}$ be a martingale with mean S_0 and T a stopping time satisfying (a) $T < \infty$ (b) $E(|S_T|) < \infty$ & (c) $E[S_n I(T \geq n)] \to 0$. Show $E(S_T) = S_0$. Does this result apply to the Gambler's Ruin problem?

3. In the $\mathcal{L}_2$ context & under appropriate assumptions prove the Kolmogorov Inequality, KHR extension, the martingale convergence theorem & a SLLN for 0-mean martingales.

4. $\{S_n, n \geq 1\}$ is called a reverse martingale if $E|S_n| < \infty$ and $E(S_n | S_{n+1}, S_{n+2}, \ldots) = S_{n+1}$. Suppose X_i are iid with mean μ. Show $\left\{\frac{X_1 + \cdots + X_n}{n}, n \geq 1\right\}$ is a reverse martingale.

5. Let $E|X_n| < \infty$ & $X_n \xrightarrow{\mathcal{L}_1} X$. Let $P(X = a) = 0$. (i) Show $E[|X_n| I(|X_n| < a)] \to E(|X| I(|X| < a))$

(ii) Show $\{X_n\}$ is ui.

6. Two gamblers A & B play successive independent games for unit stakes. The initial capital of A is a_0 & of B, b_0. Debt is not allowed. A wins against B with probability p, $0 < p < 1$ & loses with probability $q = 1 - p$. Calculate P(A ultimately loses).

7. Let $\{X_i\}$ be independent, 0-mean, in $\mathcal{L}_2$ & $\sum X_i$ converges in m.s. Show $\sum X_i$ converges a.s.

8. (a) Let $\{Y(t) \mid t \geq 0\}$ be a Yule process with $Y(0)=1$. Let T_i be the time it takes to go from a population of size i to $i+1$. Show $P(T_1 + \cdots + T_j \leq t \mid Y(0)=1) = (1-e^{-\lambda t})^j$ & calculate $P(Y(t)=j \mid Y(0)=1)$.

(b) Consider a Birth process $\{Y(t) \mid t \geq 0\}$ with $Y(0)=1$ & $\lambda_i = i^3$. Show $\exists$ a $t_0 \ni P(Y(t_0)=\infty) > 0$.

9. Let $\{Y(t) \mid t \geq 0\}$ be a pure Birth process with $Y(0) = 1$ & $\lambda_n = n\lambda$. Conditional on $Y(t) = n+1$ calculate the pdf of the time of the $n-1$ th birth.

10. (a) Let $X_1, X_2, \ldots$ be iid with mean 0 & variance σ^2. Let $S_n = \sum_{i=1}^{n} X_i$. Show $\{S_n^2 - n\sigma^2\}$ is a 0-mean martingale.

10(b) $\{W(t) \mid t \geq 0\}$ has the following properties

(i) $W(0) = 0$ (ii) $\{W(t) \mid t \geq 0\}$ has stationary independent increments

(iii) $W(t) \sim N(0, \sigma^2 t)$

Let $\lambda > 0$ and set $Y(t) = e^{-\lambda t/2} W(\lambda e^{\lambda t})$. Show that $\{Y(t) \mid t \geq 0\}$ is a stationary Gaussian process. Now let $r > 0$ and set $X_n = Y(nr)$. Verify that $\{X_n\}$ is a time homogeneous process &

$$f(x_1, x_2, \dots, x_N) = f(x_1) f(x_2 \mid x_1) f(x_3 \mid x_2) \cdots f(x_N \mid x_{N-1})$$

Note $f(x_1, x_2, \dots, x_k)$ is the joint pdf of $X_1, \dots, X_k$.