Let $X \geq 0$ be integer valued. We say it is aperiodic or non-arithmetic if it is not confined to the grid $\{nd\}$ where $d > 1$. If it were then $\frac{X}{d}$ would be non-arithmetic. Another way of saying this is to require

$$\gcd \{ n \mid P(X=n) > 0 \} = 1$$

Suppose $\{ n \mid P(X=n) > 0 \} = \{ n_1, n_2, \cdots \}$.

<u>Lemma</u> $\gcd \{ n_1, n_2, \cdots \} = 1 \Rightarrow \exists$ a finite set $\{ n_1, \cdots, n_k \}$ with $\gcd = 1$.

<u>Proof</u> Let $N = \gcd \{ n_1, \cdots, n_\ell \}$. Then N is decreasing (nonincreasing) and hence has a limit. The limit must be in $\mathbb{N}$. Suppose it is i. Then $\forall\, \ell$ large enough $N = i$ so that i is a common factor of $\{n_1, n_2, \cdots\}$. Hence $i = 1$ and there is a finite collection of n's with gcd 1.

<u>qed</u>

Another simple proof of this Lemma starts with the smallest $\neq 0$ element of $\{n_1, n_2, \dots\}$. Factor it into primes $p_1, \dots, p_k$ say. Now select one of the n's for which p_1 is not a factor. Select another for which p_2 isn't. This yields $k+1$ #'s with gcd 1.

<u>Lemma</u> Let $m_1, \dots, m_k \in \mathbb{N}$ be such that $\gcd\{m_1, \dots, m_k\} = 1$. Then $\exists$ integers $l_1, \dots, l_k$ such that $l_1 m_1 + \dots + l_k m_k = 1$

<u>Proof</u> Let $S = \{\underset{\sim}{n}'\underset{\sim}{m} \mid \underset{\sim}{n} \in \mathbb{Z}^k\}$ and set

$$c = \min S \cap \mathbb{N}$$

(Obviously $S \cap N$ is not empty – take $\underset{\sim}{n} = \underset{\sim}{1}$ for example.)

If $x \in S$ then $x = \underset{\in \mathbb{Z}}{\underset{\downarrow}{q}}\, c + r$, where $0 \le r < c$. Suppose $c = \underset{\sim}{l}'\underset{\sim}{m}$. Then we have

$$\underset{\sim}{n}'\underset{\sim}{m} = q(\underset{\sim}{l}'\underset{\sim}{m}) + r$$

$$\Rightarrow r = (\underset{\sim}{n} - q\underset{\sim}{l})'\underset{\sim}{m} \in S.$$

Hence $r = 0$ (since $r \geq 0$ & $r < c = \min S \cap N$).

$\therefore$ $x = gc$ so that c divides x and each of $m_1, \ldots, m_k$. But $\gcd\{m_1, \ldots, m_k\} = 1$ so that $c = 1$

gcd

<u>Corollary</u> Let $m_1, \ldots, m_k \in \mathbb{N}$ have $\gcd = 1$. Then $\{\underset{\sim}{n}' \underset{\sim}{m} \mid \underset{\sim}{n} \in \mathbb{Z}^k\} = \mathbb{Z}$

<u>Remark</u> We are using the notation $\underset{\sim}{n}' = (n_1, \ldots, n_k)$ and $\underset{\sim}{m}' = (m_1, \ldots, m_k)$ and so on. $\mathbb{Z}^k = \{\underset{\sim}{n}\}$. Had the gcd been $d \in \mathbb{N}$ then we would have $\underset{\sim}{\ell}' \underset{\sim}{m} = d$.

Let $X \in \mathbb{Z}^+$ be aperiodic (non-arithmetic) in the sense that X is not confined to a lattice $\{md : m = 0, 1, \cdots\}$ for integer $d > 1$. For such a rv $\exists$ $\kappa_1, \ldots, \kappa_\ell$ relatively prime (ie gcd=1) with $P(X = \kappa_i) > 0$, $i = 1, \cdots, \ell$. It is then the case that $\exists$ $N > 0$ st every $m \geq N$ can be written as a positive (≥ 0) linear combination of $\kappa_1, \cdots, \kappa_\ell$ (cf Feller, Vol I, Chapt 13, section 11 - Lemma 1). Now let $X_1, X_2, \cdots$ be iid X. If $m \geq N$ we then write it as $m_1 \kappa_1 + \cdots + m_\ell \kappa_\ell$, where the m's are ≥ 0. Set $M_0 = m_1 + \cdots + m_\ell$ (≥ 1). We then have

$$P\left(\sum_{i=1}^{M_0} X_i = m\right) \geq \prod_{i=1}^{\ell} \left[P(X = \kappa_i)\right]^{m_i} > 0$$

Now, for any integer z we can find $m, n \geq N$ st $z = m - n$. Take $Y_1, Y_2, \cdots$ to be iid X & ind of the X's. We then have $N_0 \geq 1$ st

$$P\left(\sum_{j=1}^{N_0} Y_j = n\right) > 0$$

$\therefore$ for any integer z $\exists$ $M_0, N_0 \geq 1$ st

$$P\left(\sum_{i=1}^{M_0} X_i - \sum_{j=1}^{N_0} Y_j = z\right) > 0 \qquad (*)$$

<u>Proposition</u> Let $X_1, X_2, \cdots, Y_1, Y_2, \cdots$ be ≥ 0 non-arithmetic rv's which are iid and independent of some other rv $Z \in \mathbb{Z}$ (integers). Then $\exists$ rv's $M, N > 0$ such that

$$\sum_{i=1}^{M} X_i - \sum_{j=1}^{N} Y_j \overset{wp1}{=} Z$$

<u>Proof</u>: Condition on $Z = z$. Since $\sum_{i=1}^{m}(X_i - Y_i)$ is a 0-mean random walk on the integers $\exists$ $N_1 < N_2 < \cdots \ni \sum_{i=1}^{N_k}(X_i - Y_i) = 0$, $\forall k$ wp1.

Now let M_0, N_0 be as in our previous proposition. Set

$$\alpha = P\left(\sum_{i=1}^{N_k + M_0} X_i - \sum_{j=1}^{N_k + N_0} Y_j = z\right)$$

Then $\alpha > 0$ by our previous proposition and the definition of N_k. Notice α is the same for each k. Now take a subsequence $\{N_{k_\ell}\}$ of $\{N_k\}$ st $N_{k_{\ell+1}} - N_{k_\ell} > \max(M_0, N_0)$ and set $A_\ell = \left\{\sum_{i=1}^{N_{k_\ell}+M_0} X_i - \sum_{j=1}^{N_{k_\ell}+N_0} Y_j = z\right\}$, $\ell = 1, 2, \cdots$

and
$$A_\ell' = \left\{ \sum_{N_{k_\ell}+1}^{N_{k_\ell}+M_0} X_i - \sum_{N_{k_\ell}+1}^{N_{k_\ell}+N_0} Y_j = z \right\}, \quad \ell = 1, 2, \cdots .$$

We have $P(A_\ell') = \alpha$ and the A_ℓ' are independent. $\therefore$ $P(A_\ell' \text{ io}) = 1$ so that $P(A_\ell \text{ io}) = 1$. Now uncondition the Z to get

$$P(A_\ell \text{ io}) = \sum_z P[\{A_\ell \text{ io}\} | Z = z] \, P(Z = z)$$
$$= \sum_z P(Z = z) = 1,$$

since the first part of the proof assumed $Z = z$ and actually showed $P[\{A_\ell \text{ io}\} | Z = z] = 1$.

qed

<u>Remark</u> This result shows that $P(T < \infty) = 1$ for the T in the coupling proof of the renewal theorem.